FUNCTIONAL ANALYSIS

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Contents

Preface

This book covers the class of Banach algebras that arise naturally out of the cohomology theory for Banach algebras, the algebraic version of which was developed by G. Hochschild [67].

In [60], we realize that the systematic study of the notion of amenability has its origin in the beginning of modern measure theory in the earlier part of the twentieth century. Hausdorff [44] posed the following question: Does there exist a finitely additive set function which is invariant under certain group action? With the help of his paradox, Hausdorff was able to answer this question in the negative. Further investigations into this question led Banach and Tarski in [4] to the paradox that now carries the name Banach-Tarski paradox.

The Banach-Tarski paradox is a mathematical theorem which imply the following: "An orange can be cut into finitely many pieces, and these can be reassembled to yield two oranges of the same size as the original one"

This is an application of the Banach-Tarski paradox, whose strongest form is: Let A and B be any two bounded sets in three-dimensional space with non-empty interior. Then there is a partition of A into finitely many sets which can be reassembled to yield B.

In [35], Ghahramani and Loy introduced generalized notions of amenability with the hope that it will yield Banach algebra without bounded approximate identity which nontheless had a form of amenability. So far, however, all known approximate amenable Banach algebras have bounded approximate identities.

Chapter 1

TENSOR PRODUCTS

1.1 Introduction

In this section we study several types of tensor norm. We put a lot of emphasis on Haagerup norm. We also consider the spatial norm, the maximal norm and the cross norms.

1.2 Haagerup norm

The Haagerup norm is a very important operator space cross-norm. The motivation was the consideration of operators of the form $\phi(a) = \Sigma_{i=1}^{n} u_i a v_i$ for $a \in A$ where $u_1, ..., u_n, v_1, ..., v_n$ are some fixed elements in A [24]. These operators result from the action of $\Sigma_{i=1}^{n} u_i \otimes v_i \in A \otimes A^{op}$ on A(where A^{op} is the C*-algebra A with the reversed product). If $A \subseteq B(H)$ then for $\xi, \eta \in H$ where

$\|\xi\| = 1,\ \|\eta\| = 1$, the Cauchy-Schwarz inequality implies

$$\begin{aligned}
|\langle \phi(a)\xi, \eta\rangle| &= \left|\left\langle \sum_{i=1}^{n} u_i a v_i \xi, \eta \right\rangle\right| \\
&= \left|\left\langle \sum_{i=1}^{n} a v_i \xi, u_i^* \eta \right\rangle\right| \\
&\leq \left(\sum_{i=1}^{n} \|a v_i \xi\|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \|u_i^* \eta\|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Further, $\|a v_i \xi\| \leq \|a\| \|v_i \xi\|$ and

$$\begin{aligned}
\sum_{i=1}^{n} \|v_i \xi\|^2 &= \sum_{i=1}^{n} \langle v_i \xi, v_i \xi\rangle \\
&= \sum_{i=1}^{n} \langle \xi, v_i^* v_i \xi\rangle \\
&\leq \left\|\sum_{i=1}^{n} v_i^* v_i\right\| \|\xi\|^2.
\end{aligned}$$

Similarly,

$$\sum_{i=1}^{n} \|u_i^* \eta\|^2 \leq \left\|\sum_{i=1}^{n} u_i u_i^*\right\| \|\eta\|^2.$$

So,

$$|\langle \phi(a)\xi, \eta\rangle| \leq \|a\| \left\|\sum_{i=1}^{n} u_i u_i^*\right\|^{\frac{1}{2}} \left\|\sum_{i=1}^{n} \|v_i^* v_i\right\|^{\frac{1}{2}} \|\xi\| \|\eta\|.$$

Hence, $\|\phi\| \leq \| \sum_{i=1}^{n} u_i u_i^* \|^{\frac{1}{2}} \| \sum_{i=1}^{n} \|v_i^* v_i\|^{\frac{1}{2}}$.

For the reverse inclusion, we may also allow infinite (countable) sequences of u_i and v_i provided that $\sum_{i=1}^{n} u_i u_i^*$ and $\sum_{i=1}^{n} v_i^* v_i$ are norm convergent.

Therefore, the natural definition following from these considerations is

$$\|t\|_h = \inf \left\{ \left\|\sum_{i=1}^{n} u_i u_i^*\right\|^{\frac{1}{2}} \left\|\sum_{i=1}^{n} v_i^* v_i\right\|^{\frac{1}{2}} \ :\ n \in \mathbf{N},\ t = \sum_{i=1}^{n} u_i \otimes v_i \in A \otimes B \right\}.$$

We show that Haagerup norm is actually a norm. To do this, we show that it satisfies the properties of a norm.

(i) We note that $\|t\|_h = \inf \{\| \sum_{i=1}^{n} a_i \otimes b_i \|\}$. So clearly, $\|t\|_h \geq 0$ and $\|t\|_h = 0$

if and only if $t = 0$.

(ii) We show that $\forall\ \alpha \in \mathbf{K}$, $\|\alpha t\|_h = |\alpha|\|t\|_h$.

Now,

$$\begin{aligned}
\|\alpha t\|_h &= \inf\left\{\left\|\sum_{i=1}^{n}(\alpha a_i)(\alpha a_i)^*\right\|\left\|\sum_{i=1}^{n} b_i^* b_i\right\|\right\}^{\frac{1}{2}} \\
&= \inf\left\{\left\|\sum_{i=1}^{n} a_i a_i^*\right\|\left\|\sum_{i=1}^{n}(\alpha b_i^*)(\alpha b_i)\right\|\right\}^{\frac{1}{2}} \\
&= \inf\left\{|\alpha|^2\left\|\sum_{i=1}^{n} a_i a_i^*\right\|\left\|\sum_{i=1}^{n} b_i^* b_i\right\|\right\}^{\frac{1}{2}} \\
&= |\alpha| \inf\left\{\left\|\sum_{i=1}^{n} a_i a_i^*\right\|\left\|\sum_{i=1}^{n} b_i^* b_i\right\|\right\}^{\frac{1}{2}} \\
&= |\alpha|\|t\|_h.
\end{aligned}$$

(iii) If $t,\ t' \in B(H) \otimes B(H)$ then $\forall\ t = \sum_{i=1}^{n} a_{i1} \otimes b_{i1}$ and $t' = \sum_{i=1}^{n} a_{i2} \otimes b_{i2}$,

$\|t + t'\|_h \leq \|t\|_h + \|t'\|_h$.

Now,

$$\begin{aligned}
\|t + t'\|_h &= \inf\left\{\left\|\left(\sum_{i=1}^{n} a_{i1} \otimes b_{i1}\right) + \left(\sum_{i=1}^{n} a_{i2} \otimes b_{i2}\right)\right\|\right\} \\
&\leq \inf\left\{\left\|\sum_{i=1}^{n} a_{i1} \otimes b_{i1}\right\| + \left\|\sum_{i=1}^{n} a_{i2} \otimes b_{i2}\right\|\right\} \\
&\leq \inf\left\{\left\|\sum_{i=1}^{n} a_{i1} \otimes b_{i1}\right\|\right\} + \inf\left\{\left\|\sum_{i=1}^{n} a_{i2} \otimes b_{i2}\right\|\right\} \\
&= \|t\|_h + \|t'\|_h.
\end{aligned}$$

(iv) If $t,\ t' \in B(H) \otimes B(H)$ then $\|tt'\|_h \leq \|t\|_h\|t'\|_h$.

Now,

$$\begin{aligned}
\|tt'\|_h &= \inf\left\{\left\|\left(\sum_{i=1}^n a_{i1}\otimes b_{i1}\right)\left(\sum_{i=1}^n a_{i2}\otimes b_{i2}\right)\right\|\right\}\\
&\leq \inf\left\{\left\|\sum_{i=1}^n a_{i1}\otimes b_{i1}\right\|\left\|\sum_{i=1}^n a_{i2}\otimes b_{i2}\right\|\right\}\\
&\leq \inf\left\{\left\|\sum_{i=1}^n a_{i1}\otimes b_{i1}\right\|\right\}\inf\left\{\left\|\sum_{i=1}^n a_{i2}\otimes b_{i2}\right\|\right\}\\
&= \|t\|_h\|t'\|_h.
\end{aligned}$$

The upper bound therefore, is given by $\|T\| \leq \|\sum_{i=1}^n a_i \otimes b_i\|$ in terms of the Haagerup norm $\|.\|_h$ on $B(H)\otimes B(H)$. The equality holds when the operators $a_i a_i^*$ commute and $b_i^* b_i$ commute.

In the next theorem we use the following notations. For η, $\xi \in H$ we use $\eta\otimes\xi^*$ for the rank one operator on H with $(\eta\otimes\xi^*)(\theta) = \langle\theta,\xi\rangle\eta$.

Theorem 1.2.1. *For $T \in \varepsilon\ell(B(H))$, $Tx = \sum_{i=1}^n a_i x b_i$, we have*

$$\|T\| = \sup_{p_1,p_2}\left\|\sum_{i=1}^n (p_1 a_i)\otimes(b_i p_2)\right\|$$

where $p_1, p_2 \in B(H)$ are rank one projections ($p_i^2 = p_i = p_i^* \quad (i = 1, 2)$).

Proof. Let $p_1 = \xi\otimes\xi^*$ and $p_2 = \eta\otimes\eta^*$ be one dimensional projections (where η, $\xi \in H$ are unit vectors). We look at the operator

$$T_{p_1,p_2}(x) = \sum_{i=1}^n (p_1 a_i)\otimes(b_i p_2),$$

an operator with a one dimensional range. Specifically it is the operator

$$x \mapsto \langle (Tx)\eta, \xi\rangle \xi\otimes\eta^*$$

and thus a linear functional.

For this operator, $(p_1 a_i)(p_1 a_j)^*$ are commuting and so are $(b_i p_2)^*(b_j p_2)$. Hence

$$\|T_{p_1,p_2}\| = \left\|\sum_{i=1}^n (p_1 a_i)\otimes(b_i p_2)\right\|_h.$$

Alternatively, the norm of a linear functional is the same as its completely bounded norm, hence $\|T_{p_1,p_2}\| = \|T_{p_1,p_2}\|_{cb}$ = the Haagerup tensor norm for T of the form $T = \sum_{i=1}^{n} a_i \otimes b_i$.

Clearly,

$$\begin{aligned}
\|T\| &= \sup\{\|Tx\| : x \in B(H),\ \|x\| \leq 1\} \\
&= \sup\{\mathbf{R}\langle (Tx)\eta, \xi\rangle : x \in B(H),\ \|x\| \leq 1,\ \eta,\ \xi \in H, \|\xi\| = \|\eta\| = 1\} \\
&= \sup\{\mathbf{R}\langle (T_{p_1,p_2}x)\eta, \xi\rangle : x \in B(H),\ \|x\| \leq 1,\ \eta,\ \xi \in H, \|\xi\| = \|\eta\| = 1\} \\
&= \sup_{p_1,p_2} \|T_{p_1,p_2}\|.
\end{aligned}$$

Since $\|T_{p_1,p_2}\| \leq \|T\|$, then

$$\|T\| = \sup_{p_1,p_2} \left\| \sum_{i=1}^{n} (p_1 a_i) \otimes (b_i p_2) \right\|.$$

□

Definition 1.2.2. Let A, B be normed linear spaces. The projective tensor norm, p, is a norm on $A \otimes B$ and

(i) $p(\mu) \geq w(\mu)$, $\mu \in A \otimes B$, w is a weak tensor norm.

(ii) $p(a \otimes b) = \|a\|\|b\|$, $\forall\ a \in A$ and $b \in B$.

Proof. See Bonsall [9] for proof. □

1.3 Tensor product of Hilbert Space Operators

We dedicate this section to showing that convergence of sequences of Hilbert space operators is preserved by tensor product and the converse holds in case of convergence to zero under the semigroup assumption. In particular, unlike ordinary product of operators, weak convergence is preserved by tensor product.

It is also shown that a tensor product of operators is a unilateral shift if and only if it coincides with a tensor product of a unilateral shift and an isometry. These results lead to a decomposition of a tensor product of contractions into an orthogonal direct sum of tensor products of class J_0 , strongly stable tensor products, unilateral shift tensor products, and a unitary tensor product.

Let H and K be nonzero complex Hilbert spaces. We shall consider the concept of tensor product space in terms of the single tensor product of vectors as a conjugate bilinear functional on the Cartesian product of H and K. The single tensor product of $x \in H$ and $y \in K$ is a conjugate bilinear functional $x \otimes y : H \times K \to \mathbb{C}$ defined by $(x \otimes y)(u, v) = \langle x, u\rangle\langle y, v\rangle$ for every $(u, v) \in H \times K$. The collection of all (finite) sums of single tensors $x_i \otimes y_i$ with $x_i \in H$ and $y_i \in K$ denoted by $H \otimes K$, is a complex linear space equipped with an inner product $\langle , \rangle : (H \otimes K) \times (H \otimes K) \to \mathbb{C}$ defined, for arbitrary $\Sigma_{i=1}^{n} x_i \otimes y_i$ and $\Sigma_{j=1}^{m} w_i \otimes z_i$ in $H \otimes K$ by

$$\langle \Sigma_{i=1}^{n} x_i \otimes y_i, \Sigma_{j=1}^{m} w_i \otimes z_i\rangle = \Sigma_{i=1}^{n} \Sigma_{j=1}^{m} \langle x_i, w_j\rangle\langle y_i, z_j\rangle.$$

The tensor product on $H \otimes K$ of two operators $T \in B(H)$ and $S \in B(K)$ is the operator $T \otimes S : H \otimes K \to H \otimes K$ defined by $(T \otimes S)\Sigma_{i=1}^{n} x_i \otimes y_i = \Sigma_{i=1}^{n} T x_i \otimes S y_i$ for every $\Sigma_{i=1}^{n} x_i \otimes y_i \in H \otimes K$ which lies in $B(H \otimes K)$.

The completion of the inner product space $H \otimes K$, denoted by $H \widehat{\otimes} K$, is the tensor product space of H and K. The extension of $T \otimes S$ over the Hilbert space $H \widehat{\otimes} K$, denoted by $T \widehat{\otimes} S$, is the tensor product of T and S on the tensor product space, which lies in $B(H \widehat{\otimes} K)$. For an expository paper containing the essential properties of tensor products needed here. It is exhibited in Theorem a decomposition of a tensor product contraction $T \widehat{\otimes} S$ into an orthogonal direct sum of tensor products of class J_0, strongly stable tensor products, unilateral shift tensor products, and a unitary tensor product. This is done after showing in Theorem 1.3.1 that weak, strong and uniform convergences are preserved by tensor product. The case of convergence to zero is considered in Theorem 1.3.3, and the converse is investigated under the semigroup assumption (i.e., for power sequences). The above mentioned decomposition is also based on Lemma 1, which ensures that a tensor product is a unilateral shift if and only if it coincides with a tensor product of a unilateral shift and an isometry.

1.3.1 Convergence of sequences of operators

A sequence $\{T_n\}$ of operators in $B(H)$ converges uniformly, or strongly, or weakly to an operator $T \in B(H)$ if $\|T_n - T\| \to 0$, or $\|(T_n - T)x\| \to 0$ for every $x \in H$, or $\langle T_n x, y\rangle \to 0$ for every $x, y \in H$ (equivalently, $\langle T_n x, x\rangle \to 0$ for every x in the complex Hilbert space H), and these will be denoted by $T_n \to^u T$, or $T_n \to^s T$, or $T_n \to^w T$, respectively. It is bounded if $\sup_n \|T_n\| < \infty$. Clearly,

$$T_n \to^u T \Rightarrow T_n \to^s T \Rightarrow T_n \to^w T \Rightarrow \sup_n \|T_n\| < \infty.$$

Theorem 1.3.1. *Let $\{T_n\}$ and $\{S_n\}$ be sequences of operators in $B(H)$ and $B(K)$ respectively. Let also $T \in B(H)$ and $S \in B(K)$.*

(a) If $T_n \to^u T$ and $S_n \to^u S$, then $T_n \widehat{\otimes} S_n \to^u T \widehat{\otimes} S$.

(b) If $T_n \to^s T$ and $S_n \to^s S$, then $T_n \widehat{\otimes} S_n \to^s T \widehat{\otimes} S$.

(c) If $T_n \to^w T$ and $S_n \to^w S$, then $T_n \widehat{\otimes} S_n \to^w T \widehat{\otimes} S$.

Proof. Recall that $T_n \otimes S_n - T \otimes S = T_n \otimes (S_n - S) + (T_n - T) \otimes S$ for each n , which still holds if $\otimes$ is replaced with $\widehat{\otimes}$.

(a) If $\|T_n - T\| \to 0$ (so that $\{T_n\}$ is bounded) and $\|S_n - S\| \to 0$, then $\|T_n \widehat{\otimes} S_n - T \widehat{\otimes} S\| \leq \sup_n \|T_n\| \|S_n - S\| + \|S\| \|T_n - T\|$, and hence then $\|T_n \widehat{\otimes} S_n - T \widehat{\otimes} S\| \to 0$. That is $T_n \widehat{\otimes} S_n \to^u T \widehat{\otimes} S$.

(b) Take an arbitrary $\Sigma_{i=1}^n x_i \otimes y_i$ in $H \otimes K$ and observe that

$\|T_n \otimes S_n - T \otimes S \Sigma_{i=1}^n x_i \otimes y_i\| \leq \sup_n \|T_n\| \Sigma_{i=1}^n \|x_i\| \Sigma_{i=1}^n \|(S_n - S) y_i\|$

$+ \|S\| \Sigma_{i=1}^n \|y_i\| \Sigma_{i=1}^n \|(T_n - T) x_i\|$.

If If $T_n \to^s T$ and $S_n \to^s S$, then $\|(T_n \otimes S_n - T \otimes S) \Sigma_{i=1}^n x_i \otimes y_i\| \to 0$ and so $T_n \otimes S_n \to^s T \otimes S$. Moreover, $\{T_n \widehat{\otimes} S_n\}$ is bounded (because $\sup_n \|T_n \widehat{\otimes} S_n \leq \sup_n \|T_n\| \sup_n \|S_n\| < \infty$). As it is well-known, if a sequence of operators converges strongly in a normed space, and if its extension is bounded in the completion, then convergence holds in the completion of the space. Thus $T_n \widehat{\otimes} S_n \to^s T \widehat{\otimes} S$.

(c) Similarly, and applying the Schwarz inequality,

$|\langle T_n \otimes S_n - T \otimes S \Sigma_{i=1}^n x_i \otimes y_i, \Sigma_{i=1}^n x_i \otimes y_i \rangle|$

$\leq \sup_n \|T_n\| \Sigma_{i=1}^n \Sigma_{j=1}^n \|x_i\| \|x_j\| \Sigma_{i=1}^n \Sigma_{j=1}^n |\langle (S_n - S) y_i, y_j \rangle|$

$+ \|S\| \Sigma_{i=1}^n \Sigma_{j=1}^n \|y_i\| \|y_j\| \Sigma_{i=1}^n \Sigma_{j=1}^n |\langle (T_n - T) x_i, x_j \rangle|$.

Thus $|\langle T_n \otimes S_n - T \otimes S \Sigma_{i=1}^n x_i \otimes y_i, \Sigma_{i=1}^n x_i \otimes y_i \rangle| \to 0$, whenever $T_n \to^w T$ and $S_n \to^w S$, and so $T_n \otimes S_n \to^w T \widehat{\otimes} S$. The same argument applies for weak convergence so that $T_n \widehat{\otimes} S_n \to^w T \widehat{\otimes} S$.

$\square$

Remark 1.3.2. The result of part (c) in Theorem 1.3.1 does not mirror the ordinary product counterpart. Indeed, $T_n \to^w T$ and $S_n \to^w S$ do not imply $T_n S_n \to^w TS$. In fact, even $T_n \to^s T$ and $S_n \to^w S$ do not imply $T_n S_n \to^w TS$. We give the following example. If V is a unilateral shift, then if we put $T_n^* =$

$S_n = V^n$ so that $T_n \to^s 0$, $S_n \to^w 0$, we have $T_n S_n = I$ for every n.

Theorem 1.3.3. *Let $\{T_n\}$ and $\{S_n\}$ be sequences of operators in $B(H)$ and $B(K)$ respectively. If one of them converges to zero uniformly (strongly, weakly) and the other is bounded, then $\{T_n \widehat{\otimes} S_n\}$ converges to zero uniformly (strongly, weakly).*

Proof. If $\|T_n\| \to 0$ and $\sup_n \|S_n\| < \infty$ or vice versa, then $\|T_n \widehat{\otimes} S_n\| \to 0$ because $\|T_n \widehat{\otimes} S_n\| = \|T_n \otimes S_n\| = \|T_n\|\|S_n\|$ for every $n \geq 1$, which proves the claimed result for uniform convergence. For strong and weak convergences take an arbitrary vector $\Sigma_{i=1}^n x_i \otimes y_i$ in $H \otimes K$. Note that

$$\|(T_n \otimes S_n)\Sigma_{i=1}^n x_i \otimes y_i\| \leq \sup_n \|S_n\|\Sigma_{i=1}^n \|T_n x_i\|\Sigma_{i=1}^n \|y_i\|.$$

If $\{T_n\}$ converges strongly to zero and if $\{S_n\}$ is bounded (or vice versa), then $\|(T_n \otimes S_n)\Sigma_{i=1}^n x_i \otimes y_i\| \to 0$. Applying the same argument in the proof of (b) of Theorem 1.3.1 above we get $T_n \widehat{\otimes} S_n \to^s 0$. Similarly,

$|\langle (T_n \otimes S_n)\Sigma_{i=1}^n x_i \otimes y_i, \Sigma_{i=1}^n x_i \otimes y_i\rangle|$

$\leq \sup_n \|S_n\|\Sigma_{i=1}^n \Sigma_{j=1}^n |\langle T_n x_i, x_j\rangle|\Sigma_{i=1}^n \Sigma_{j=1}^n \|y_i\|\|y_j\|$

If $\{T_n\}$ converges weakly to zero and if $\{S_n\}$ is bounded (or vice versa), then $\langle (T_n \otimes S_n)\Sigma_{i=1}^n x_i \otimes y_i, \Sigma_{i=1}^n x_i \otimes y_i\rangle \to 0$. Again, applying the same argument in the proof of (c) of Theorem 1.3.1, it follows that $T_n \widehat{\otimes} S_n$ converges weakly to zero. $\square$

At this juncture we give some results on convergence of power sequences of Hilbert space operators.We shows that, unlike the above example, convergence to zero of power sequences (or, equivalently, of sequences having the semigroup property) is transferred from the tensor product to one of the factors. First we consider the following auxiliary result.

Proposition 1.3.4. *If the power sequence* $\{T^n \widehat{\otimes} S^n\}$ *is bounded, then so is one of the power sequences* $\{T^n\}$ *or* $\{S^n\}$ *.*

Theorem 1.3.5. *Let* $T \in B(H)$ *and let* $S \in B(K)$ *. Consider the power sequences* $\{T^n\}$ *and* $\{S^n\}$*. If* $\{T^n \widehat{\otimes} S^n\}$ *converges to zero uniformly or strongly, then so does one of the sequences* $\{T^n\}$ *or* $\{S^n\}$*. If* $\{T^n \widehat{\otimes} S^n\}$ *converges to zero weakly, and one of* T *or* S *is power incremented, then one of the sequences* $\{T^n\}$ *or* $\{S^n\}$*.*

Proof. See Bonsall [9, Theorem 7.1]. □

1.4 Tensor product of C*-algebras

Theorem 1.4.1. *Let* A, B *be normed algebras over* $\boldsymbol{K}$*. There exists a unique product on* $A \otimes B$ *with respect to which* $A \otimes B$ *is an algebra and* $(a \otimes b)(c \otimes d) = ac \otimes bd$ $(a, c \in A,\ b, d \in B)$*.*

We note that $A \otimes B$ endowed with multiplication is called the **algebra tensor product** and $A \otimes B$ together with an involution the ***-algebra tensor product**.

The norm of a C*-algebra is unique in the sense that on a given *-algebra A there is at most one norm which makes A into a C*-algebra. We consider two types of norms and we determine the relationship between them.

1.4.1 Spatial norm

The norm $\|.\|_\pi$ defined by the inclusion $A \otimes B \subseteq B(H) \otimes B(K) \subseteq B(H \hat{\otimes} K)$ is called the spatial norm, assuming that A and B are faithfully represented on Hilbert spaces H and K respectively. This norm was introduced by T. Turumaru in 1953. The definition does not depend on particular representations of A and B, that is, $\forall\, t \in A \otimes B$

$$\|t\|_\pi = \|\theta \otimes \vartheta(t)\|_{B(H \otimes K)}$$

for any two faithful representations θ of A on H and ϑ of B on K.

1.4.2 Maximal C*-norm

The second natural norm on $A \otimes B$ was introduced in 1965 by A. Guichardet. It is the *maximal C^*-norm* $\|.\|_\nu$ defined as:

$$\|t\|_\nu = \sup\{\|\tau t\| : \tau \; is \; a \; subtensor \; representation \; of \; A \otimes B\}, \; for \; t \in A \otimes B.$$

1.5 Normed Spaces

We begin with the description of the well-known class of real normed linear spaces. A normed linear space is a metric space if the distance between x and y is taken as $\|x - y\|$. Let us recall the most characteristic examples of real normed linear space.

In the space $\mathbb{R}^n$, we have: $\|.\|_e$, the classical Euclidean norm, $\|.\|_+$ the so-called taxi-cab norm, $\|.\|_\vee$ the diamond norm and $\|.\|_m$ km is a mixed norm. If l^∞ denotes the linear space of all bounded real sequences then its natural norm is defined by $\|x\|_\infty = \sup\{|x_n| : n \geq 1\}$, for any bounded real sequence $x = (x_n)$. One can restrict this norm to the subspace c of all convergent real sequences and, in particular, to the subspace c_0 of all real sequences convergent to 0.

Given a compact space K, let $C(K)$ be the vector space of all real-valued continuous functions f defined on K. Then one considers the norm

$$\|f\| = \sup\{|f(x)| : x \in K\}.$$

Consider the L^p space. A closed real interval $[a, b]$ being fixed with $a < b$, for $p \geq 1$, let L^p denote the space of continuous real-valued functions defined on $[a, b]$ and such that $\int_a^b |f(t)|^p dt < \infty$. Then one defines the norm

$$\|f\|_p = (\int_a^b |f(t)|^p dt)^{\frac{1}{p}}.$$

Theorem 1.5.1. *Assume that $(V, \|.\|)$, $(W, \|.\|)$ are real normed spaces and let f be a surjective mapping from V onto W which is an isometry, i.e., $\|f(x) - f(y)\| = \|x - y\|$, for all $x, y \in V$. Then the mapping $T := f - f(0)$ is linear.*

Chapter 2

AMENABILITY IN BANACH ALGEBRAS

2.1 Introduction

The notion of an amenable Banach algebra was introduced by B.E. Johnson in his definitive monograph. This class of Banach algebras arises naturally out of the cohomology theory for Banach algebras, the algebraic version of which was developed by G. Hochschild [67].

In [60], we realize that the systematic study of the notion of amenability has its origin in the beginning of modern measure theory in the earlier part of the twentieth century. Hausdorff [44] posed the following question: Does there exist a finitely additive set function which is invariant under certain group action? With the help of his paradox, Hausdorff was able to answer this question in the negative. Further investigations into this question led Banach and Tarski in [4] to the paradox that now carries the name Banach-Tarski paradox.

The Banach-Tarski paradox is a mathematical theorem which imply the following: "An orange can be cut into finitely many pieces, and these can be reassembled to yield two oranges of the same size as the original one"

This is an application of the Banach-Tarski paradox, whose strongest form is: Let A and B be any two bounded sets in three-dimensional space with non-empty interior. Then there is a partition of A into finitely many sets which can be reassembled to yield B.

In [35], Ghahramani and Loy introduced generalized notions of amenability with the hope that it will yield Banach algebra without bounded approximate identity which nontheless had a form of amenability. So far, however, all known approximate amenable Banach algebras have bounded approximate identities. They gave examples to show that for most of these new notions, the corresponding class of Banach algebras is larger than that for the classical amenable Banach algebra introduced by Johnson in [47]. They developed general theory for these notions, and studied them for several concrete classes of Banach algebra.

Let A be a Banach algebra over $\mathbf{C}$ and $\varphi : A \to \mathbf{C}$ be a character on A, that is, an algebra homomorphism from A into $\mathbf{C}$, and let Φ_A denote the character space of A (that is the set of all characters on A). In [55], Monfared introduced the notion of character amenable Banach algebras. His definition of this notion requires continuous derivations from A into dual Banach A-bimodules to be inner, but only those modules are concerned where either of the left or right module action is defined by characters on A. As such character amenability is weaker than the classical amenability introduced by Johnson in [48], so all amenable Banach algebras are character amenable.

In this chapter, we shall continue in the spirit as in [35] and [55] by applying the concept of approximate amenability to that of character amenability and introduced the notions of approximate left character amenability, approximately right character amenability and approximately character amenability. We develop general theory on these notions and study them for Banach algebras defined over locally compact groups and second duals of Banach algebras.

2.2 Modules and bimodules

If A is a normed algebra and X is a normed space, then X is said to be a normed left A-module if X is a left A-module and there exists a positive constant k such that $\|ax\| = k\|a\|\|x\|$. The notion of a normed right A-module is similarly defined. A space X is said to be an A-bimodule, if it is both a normed left A-module and a normed right A-module and $(ax)b = a(xb)(a,\ b \in A,\ x \in X)$. Furthermore, if X is a complete space, then it is called a Banach module. Let A be a Banach algebra and let X be a Banach A-bimodule. Then the dual of X, X' is a also a Banach A-bimodule. Indeed the module action is defined by:

$$\langle x, a.\alpha\rangle = \langle x.a, \alpha\rangle \;\; (a \in A,\ , \alpha \in X'x \in X)$$

$$\langle x, \alpha.a\rangle = \langle a.x, \alpha\rangle \;\; (a \in A,\ , \alpha \in X'x \in X)$$

Similarly, the higher duals $X^{(n)}$ are Banach A-bimodules. If the canonical embedding of X in X'' is denoted by $\hat{x}$ then $a.\hat{x} = \widehat{a.x}$, so that $X^{(n)}$ is a submodule of $X^{(n+2)}$ for each $n \in Z$.

Definition 2.2.1. A derivation from A into X is a linear map D such that

$$D(ab) = a.D(b) + D(a).b \;\; (a, b \in A).$$

For example, for any $x \in X$, the mapping $\delta_x : A \to X$ given by

$\delta_x(a) = ax - xa$; $(a \in A)$ is a derivation, called an inner derivation.

The space of continuous derivations from A into X is denoted by $Z^1(A, X)$ and the space of inner derivations from A into X is denoted by $N^1(A, X)$. The first cohomology group of A with coefficients in X is defined by

$$H^1(A, X) = \frac{Z^1(A, X)}{N^1(A, X)}.$$

The Banach algebra A is amenable if $H^1(A, X') = \{0\}$ for each Banach A-bimodule X; and it is weakly amenable if $H^1(A, A') = \{0\}$. Let A be a Banach algebra and let I be a closed ideal of A, A is I-weakly amenable if $H^1(A, I') = \{0\}$. A is $n - I$-weakly amenable if $H^1(A, I^{(n)}) = \{0\}$. A is ideally amenable if A is I-weakly amenable for every closed ideal I in A. A Banach algebra A is amenable if every continuous derivation $D : A \to X'$ is inner for every Banach A-bimodule X. In particular, if G is a locally compact group then $L^1(G)$ is amenable (as a Banach algebra) if and only if G is amenable as a topological group.

2.2.1 Key examples of modules

Example 2.2.2. A is an A-bimodule when the left and right actions are defined by the multiplication of A in a usual manner.

Example 2.2.3. If X is an A-bimodule then the dual X' of X is also an A-bimodule under the actions

$$(af)(x) = f(xa), (fa)(x) = f(ax)(a \in A,\ x \in X).$$

Definition 2.2.4. Let A be a complex algebra. If $I = \mathbb{K}a$ is a left ideal of A for each $a \in A$, then A is called an ideally factored algebra (IF-Algebra). Each multiple ba of a is therefore of the form $\phi(b)a$ for some $\phi(b) \in \mathbb{K}$. This ensures us to have a simple formula for the multiplication and helps us to consider problems concerning linear mappings such as homomorphisms and derivations involving this type of multiplication.

The multiplication on an IF-Algebra is interesting on its own right. Therefore, we need to characterize all ideally factored algebras and to investigate their general properties.

Definition 2.2.5. Let A be a complex Banach space and $\phi : A \to \mathbb{C}$ be a bounded linear functional. Define a multiplication on A by $ab = \phi(a)b$. This multiplication evidently makes A into a Banach algebra denoted by A_ϕ which is called the IF-Algebra associated to ϕ.

Proposition 2.2.6. *Let A_ϕ be an IF-algebra. Then there is a multiplicative linear functional $\phi : A \to \mathbb{C}$ such that $ba = \phi(b)a$ for all $b \in A_\phi$.*

Proof. For each $a \in A_\phi$, $I_a = \mathbb{C}a$ is a left ideal. Thus, for each $b \in A_\phi$ there is a $\phi_a(b) \in \mathbb{C}$ such that $ba = \phi_a(b)a$. Clearly, ϕ_a is linear. Since the multiplication of A_ϕ is associative, we have

$$\phi_a(b)\phi_a(c)a = b(\phi_a(c)a) = b(ca) = (bc)a = \phi_a(bc)a.$$

Hence, ϕ_a is multiplicative. Next, we show that $\phi_a = \phi_b$ for all $a,\ b \in A_\phi$. If $\{a, b\}$ is a linearly independent set, then for each $c \in A_\phi$ we have

$$\phi_{a+b}(c)(a + b) = c(a + b) = ca + cb = \phi_a(c)a + \phi_b(c)b.$$

Therefore, $(\phi_{a+b} - \phi_a)(c)a = (\phi_b - \phi_{a+b})(c)b$. Now, since $\{a, b\}$ is a linearly independent, we have that

$$(\phi_{a+b} - \phi_a)(c) = (\phi_b - \phi_{a+b})(c) = 0.$$

Hence, $\phi_a = \phi_{a+b} = \phi_b$. If $\{a, b\}$ is not linearly independent, say $b = \beta a$ for a nonzero scalar β, then

$$\phi_{\beta a}(c)\beta a = c(\beta a) = \beta(ca) = \beta\phi_a(c)a.$$

Thus, $\phi_{\beta a}(c) = \phi_a(c)$ and so $\phi_b = \phi_{\beta a} = \phi_a$. $\square$

Next we give some results on the general properties of A_ϕ.

Definition 2.2.7. Let A is a complex normed algebra with an identity and $a \in A$. We define the spectrum $\sigma_{A_\phi}(a)$ of a by

$$\sigma_{A_\phi}(a) = \{\lambda \in \mathbb{C} : a - \lambda I \textit{ is not invertible}\}$$

and the spectral radius $r_{A_\phi}(a)$ of a is defined by

$$r_{A_\phi}(a) = \sup\{|\lambda| : \lambda \in \sigma_{A_\phi}(a)\} = \lim_n \|a^n\|^{\frac{1}{n}}.$$

If A has no identity, then the spectrum of $a \in A$ is determined by its spectrum in the unitization $A^+ = A \oplus \mathbb{C}$ of A.

Theorem 2.2.8. *The Banach algebra A_ϕ has the following properties:*

(i) ϕ *is multiplicative.*

(ii) $u \in A_\phi$ *is a left identity if and only if* $\phi(u) = 1$.

(iii) $u \in A_\phi$ *is a right identity if and only if* A_ϕ *is linearly spanned by* u. *itself.*

(iv) $(a, \mu) \in A^+_\phi$ *is invertible if and only if* $\mu \neq 0$ *and* $\phi_a \neq -\mu$ *where,* $(a, \mu)^{-1} = \left(\frac{-a}{\mu(\phi(a)+\mu)}, \frac{1}{\mu}\right)$.

(v) $\sigma_{A_\phi}(a) = \{0, \phi(a)\}$.

(vi) $r_{A_\phi}(a) = |\phi(a)|$.

(vii) ϕ is injective if and only if A_ϕ is commutative.

Proof. (i) $\phi(ab) = \phi(\phi(a)b) = \phi(a)\phi(b)$.

(ii) $b = ub = \phi(u)b$ for all $b \in A_\phi$ if and only if $\phi(u) = 1$.

(iii) $b = bu = \phi(b)u$ if and only if each element of A_ϕ is a scalar multiple of u.

(iv) We have that

$$\begin{aligned}(a,\mu)(b,\lambda) &= (a.b + \lambda a + \mu b, \mu\lambda)\\ &= (\phi(a)b + \lambda a + \mu b, \mu\lambda)\\ &= ((\phi(a) + \mu)b + \lambda a, \mu\lambda).\end{aligned}$$

Therefore, $(a,\mu)(b,\lambda) = (0,1)$ if and only if $b = \frac{-1}{\mu(\phi(a)+\mu)}a$ and $\lambda = \frac{1}{\mu}$.

(v) By (iii), $(a,0) - \lambda(0,1) = (a,-\lambda)$ is invertible if and only if $\phi(a) \neq 0$ and $\lambda \neq 0$. Hence, $\sigma_{A_\phi}(a,0) = \{\phi(a), 0\}$.

(vi) $r_{A_\phi}(a) = \max\{|\lambda| : \lambda \in \sigma_{A_\phi}(a)\} = |\phi(a)|$.

(vii) Let ϕ be one to one, then

$$\phi(ab) = \phi(\phi(a)b) = \phi(a)\phi(b) = \phi(b)\phi(a) = \phi(\phi(b)a) = \phi(ba).$$

Therefore, $ab = ba$. Conversely, let $\phi(a) = \phi(b)$ for some $a, b \in A_\phi$. Since A_ϕ is commutative, $ab = ba$. So, $\phi(a)b = \phi(b)a$. Hence $a = b$. Therefore, ϕ is one to one. $\square$

Let A be an algebra. The character space of A is denoted by Φ_A. Let X be an A-bimodule. A *derivation* from A to X is a linear map $D : A \to X$ such that

$$D(ab) = Da \cdot b + a \cdot Db \quad (a, b \in A)\,.$$

For example, $\delta_x : a \to a \cdot x - x \cdot a$ is a derivation; derivations of this form are the *inner derivations.*

Let X be a Banach space, the second dual of X is X'', and the canonical embedding of X in X'' is denoted by i. The image of X in X'' under i is denoted by $\widehat{X}$. The weak-* topology on X' is denoted by $\sigma(X', X)$. We shall use the Goldstine's theorem: for each $\Lambda \in X''$, there is a net x_α in X such that $\|x_\alpha\| \leq \|\Lambda\|$ and $x_\alpha \to \Lambda$ in $(X'', \sigma(X'', X'))$.

Let A be a Banach algebra, and let X be an A-bimodule. Then X is a Banach A-bimodule if X is a Banach space and if there is a constant $k > 0$ such that

$$\|a \cdot x\| \leq k\|a\|\|x\|, \quad \|x \cdot a\| \leq k\|a\|\|x\| \quad (a \in A,\, x \in X)\,.$$

By renorming X, we can suppose that $k = 1$. For example, A itself is Banach A-bimodule, and X', the dual space of a Banach A-bimodule X, is a Banach A-bimodule with respect to the module operations specified for by

$$\langle x,\, a \cdot \lambda\rangle = \langle x \cdot a,\, \lambda\rangle, \quad \langle x,\, \lambda \cdot a\rangle = \langle a \cdot x,\, \lambda\rangle \quad (x \in X)$$

for $a \in A$ and $\lambda \in X'$; we say that X' is the *dual module* of X.

Let A be a Banach algebra, and let X be a Banach A-bimodule. Then $\mathcal{Z}^1(A, X)$ is the space of all continuous derivations from A into X, $\mathcal{N}^1(A, X)$ is the space of all inner derivations from A into X, and the first cohomology group of A with coefficients in X is the quotient space

$$\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X)/\mathcal{N}^1(A, X)\,.$$

The Banach algebra A is *amenable* if $\mathcal{H}^1(A, X') = \{0\}$ for each Banach A-bimodule X.

A derivation $D : A \to X$ is *approximately inner* if there is a net (x_v) in X such that

$$D(a) = lim_v(a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in A),$$

the limit being taken in $(X, \|.\|)$. That is, $D(a) = lim_v \delta_{x_v}(a)$, where (δ_{x_v}) is a net of inner derivations. The Banach algebra A is *approximately amenable* if, for each Banach A-bimodule X, every continuous derivation $D : A \to X'$ is approximately inner.

We let $\mathcal{M}^A_{\varphi_r}$ denote the class of Banach A- bimodule X for which the right module action of A on X is given by $x \cdot a = \varphi(a)x \quad (a \in A, x \in X, \varphi \in \Phi_A)$, and $\mathcal{M}^A_{\varphi_l}$ denote the class of Banach A- bimodule X for which the left module action of A on X is given by $a \cdot x = \varphi(a)x \quad (a \in A, x \in X, \varphi \in \Phi_A)$. If the right module action of A on X is given by $x \cdot a = \varphi(a)x$, then it is easy to see that the left module action of A on the dual module X' is given by $a \cdot f = \varphi(a)f \quad (a \in A, f \in X', \varphi \in \Phi_A)$. Thus, we note that $X \in \mathcal{M}^A_{\varphi_r}$ (resp. $X \in \mathcal{M}^A_{\varphi_l}$) if and only if $X' \in \mathcal{M}^A_{\varphi_l}$ (resp. $X' \in \mathcal{M}^A_{\varphi_r}$).

Let A be a Banach algebra and let $\varphi \in \Phi_A$, we see [55] that:

(i) A is left φ-amenable if every continuous derivation $D : A \to X'$ is inner for every $X \in \mathcal{M}^A_{\varphi_r}$;

(ii) A is right φ-amenable if every continuous derivation $D : A \to X'$ is inner for every $X \in \mathcal{M}^A_{\varphi_l}$;

(iii) A is left character amenable if it is left φ-amenable for every $\varphi \in \Phi_A$;

(iv) A is right character amenable if it is right φ-amenable for every $\varphi \in \Phi_A$;

(v) A is character amenable if it is both left and right character amenable.

Chapter 3

APPROXIMATE AMENABILITY

3.1 General theory

We introduce the following definitions.

Definition 3.1.1. Let A be a Banach algebra and let $\varphi \in \Phi_A$.Then we say that

(i) A is approximately left φ-amenable if every continuous derivation

$D : A \to X'$ is approximately inner for every $X \in \mathcal{M}^A_{\varphi_r}$;

(ii) A is approximately right φ-amenable if every continuous derivation

$D : A \to X'$ is approximately inner for every $X \in \mathcal{M}^A_{\varphi_l}$;

(iii) A is approximately left character amenable if it is approximately left φ-amenable for every $\varphi \in \Phi_A$;

(iv) A is approximately right character amenable if it is approximately right φ-amenable for every $\varphi \in \Phi_A$;

(v) A is approximately character amenable if it is both approximately left and approximately right character amenable.

Clearly, approximate character amenability is weaker than character amenability, approximate amenability, and amenability, and so, every character amenable, approximately amenable and amenable Banach algebra is approximately character amenable.

Definition 3.1.2. Let A be a Banach algebra and $\varphi \in \Phi_A$. A left [right] approximate φ-mean is a net $(m_v) \subset A''$ such that

(i) $m_v(\varphi) = 1$;

(ii) $\|\varphi(a)m_v - m_v \cdot a\| \to 0$ $[\|a \cdot m_v - \varphi(a)m_v\| \to 0]$ $\quad (a \in A)$.

Theorem 3.1.3. *Let A be a Banach algebra and $\varphi \in \Phi_A$. Then the following statements are equivalent:*

(i) A has a left [right] approximate φ-mean;

(ii) A is left [right] approximately φ-amenable;

(iii) Given $(ker\varphi)''$ a dual A-bimodule structure by taking left [right] action to be $a \cdot m = \varphi(a)m$ $\;[m \cdot a = \varphi(a)m]$ $\quad (a \in A, m \in A'')$ and taking the right [left] action to be the natural one. Then any continuous derivation $D : A \to (ker\varphi)''$ is approximately inner.

Proof. The equivalence of (i) and (ii) can be shown as in the classical case of φ-amenable Banach algebra by simple modifications of the arguments used in the proof of [**?**, Theorem 1.1]. Trivially, (ii) implies (iii). We only have to proof (iii) $\Rightarrow$ (i). For this, let $b \in B$ be such that $\varphi(b) = 1$, and let $D : A \to (ker\varphi)''$ be defined by $D(a) = \varphi(a)b - ba, \quad (a \in A)$. It is easy to see that D is a derivation, and so it is approximately inner by (iii). Thus there exists a net (n_v) in $(ker\varphi)''$

with

$$D(a) = lim_v(\varphi(a)n_v - n_v \cdot a) \quad (a \in A)).$$

That is,

$$\varphi(a)b - ba = lim_v(\varphi(a)n_v - n_v \cdot a) \quad (a \in A)) \qquad (3.1).$$

Set $m_v = b - n_v$. Then clearly $m_v(\varphi) = 1$ and

$$\|\varphi(a)m_v - m_v \cdot a\| = \|\varphi(a)b - ba - (\varphi(a)n_v - n_v \cdot a)\| \to 0 \quad (a \in A)$$

by (3.1). Thus A has a left aproximate φ-mean. The right version of (iii) $\Rightarrow$ (i) is similar. □

Proposition 3.1.4. *Let A be a Banach algebra.*

(i) Let $\varphi \in \Phi_A$ and suppose A is left [right] approximately φ-amenable. Then A has a left [right] approximate identity.

(ii) Suppose that A is approximately left [right] character amenable. Then A has a left [right] approximate identity.

(iii) Suppose that A is approximately character amenable. Then A has left and right approximate identity.

Proof. Let $X = A''$ with the left action as $a \cdot m = \varphi(a)m, a \in A, m \in A''$ and zero right action. Then the natural injection $i : A \to A''$, defined by $i(a) = \widehat{a} \quad (a \in A)$ is a derivation. Thus, since A is approximately left φ-amenable, there is a net (n_v) in A'' with

$$i(a) = lim_v(a \cdot n_v - n_v \cdot a) = lim_v \varphi(a)n_v.$$

That is $\varphi(a)n_v \to \widehat{a}$ for each $a \in A$. Take finite sets $F \subset A, G \subset A'$ and $\epsilon > 0$. Let $H = \{\varphi(a)\lambda : a \in A, \lambda \in G\}$. Then there is $u = u(F, G, \epsilon)$ such that

$\|\widehat{a}-\varphi(a)n_v\| < \frac{\epsilon}{2^k}$ for $a \in F$. By Goldstine's theorem, there is $b_v \in A$ such that

$$|\langle \Psi, b_v\rangle - \langle n_v, \Psi\rangle| < \frac{\epsilon}{2} \quad (\Psi \in H).$$

Thus for each $a \in F, \lambda \in G$, we have

$$|\langle \lambda, b_v a\rangle - \langle \lambda, a\rangle| \leq |\langle \lambda, b_v a\rangle - \langle \varphi(a)n_v, \lambda\rangle| + |\langle \varphi(a)n_v - \widehat{a}, \lambda\rangle|$$

$$\leq |\langle \varphi(a)\lambda, b_n\rangle - \langle n_v, \varphi(a)\lambda\rangle| + \frac{\epsilon}{2} \leq \epsilon.$$

Thus $(b_v)_{F,G,\epsilon}$ is a weak left approximate identity for A. Since A has a weak left (or right) approximate identity imply A has a left (or right) approximate identity [35, Lemma 2.1]. Then we conclude that A has a left approximate identity. An analogous argument works on the right. (ii) and (iii) follows from (i). $\square$

Let A be a Banach algebra, $A\hat{\otimes}A$ denotes the projective tensor product and $\pi : A\hat{\otimes}A \to A$ defined by $\pi(a \otimes b) = ab \quad (a, b \in A)$ be the diagonal operator. Then $A\hat{\otimes}A \to A$ is Banach A-bimodule with the module actions

$a \cdot (b \otimes c) = ab \otimes c, (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$

Theorem 3.1.5. *Let A be a Banach algebra and $\varphi \in \Phi_A$. A is left [right] approximately φ-amenable if and only if either of the following equivalent conditions holds:*

(i) there is a net $(M_v) \subset (A\hat{\otimes}A)''$ such that for each $a \in A$

$$\varphi(a)M_v - M_v \cdot a \to 0 \quad [a \cdot M_v - \varphi(a)M_v \to 0]$$

and $\langle M_v, \varphi \otimes \varphi\rangle = \pi''(M_v)(\varphi) \to 1$;

(ii) there is a net $(M'_v) \subset (A\hat{\otimes}A)''$ such that for each $a \in A$

$$\varphi(a)M'_v - M'_v \cdot a \to 0 \quad [a \cdot M'_v - \varphi(a)M'_v \to 0]$$

and $\langle M_v', \varphi\otimes\varphi\rangle = \pi''(M_v')(\varphi) = 1.$

Proof. Suppose that A is approximately left φ-amenable. Consider the Banach A-bimodule $A\hat{\otimes}A$ with the module actions given by

$a\cdot(b\otimes c) = \varphi(a)b\otimes c, (b\otimes c)\cdot a = b\otimes ca \quad (a,b,c\in A)$. Consider the quotient Banach A-bimodule $X = (A\hat{\otimes}A)'/\mathbf{C}\cdot(\varphi\otimes\varphi)$. Let $u\in(A\hat{\otimes}A)''$ be such that $u(\varphi\otimes\varphi) = 1$, and let $\delta_u : A\to(A\hat{\otimes}A)''$ be the inner derivation by u. Then the image of δ_u is a subset of $X' = \{\varphi\otimes\varphi\}^\circ$, and since A is approximately left φ-amenable, there is a net (u_v) in X' such that $\delta_u(a) = lim_v\delta_{u_v}(a)$. That is

$$\varphi(a)u - u\cdot a = lim_v(\varphi(a)u_v - u_v\cdot a) \qquad (3.2).$$

Set $M_v' = u - u_v$. Then for $a\in A$, we have

$$\varphi(a)M_v' - M_v'\cdot a = \varphi(a)u - u\cdot a - (\varphi(a)u_v - u_v\cdot a)\to 0$$

by (3.2). Also

$$\pi''(M_v')(\varphi) = \pi''(u-u_v)(\varphi) = \langle u, \varphi\otimes\varphi\rangle = 1.$$

Thus we have shown (ii).

Now suppose (i) holds. Let $X\in\mathcal{M}_{\varphi_r}^A$ and $D : A\to X'$ be a derivation. We follow the standard argument in [35, Theorem 2.1] with due care given to the module actions defined on X and X'. For each v, set $f_v(x) = M_v(\mu_x)$, where $a,b\in A, x\in X$, $\mu_x(a\otimes b) = (\varphi(a)D(b))(x)$. Then with $(m_v^\alpha)\subset A\hat{\otimes}A$ converging weak* to M_v, and noting that for $m\in A\hat{\otimes}A$,

$$\mu_{\varphi(a)x - x\cdot a}(m) = (\varphi(a)\mu_x - a\cdot\mu_x)(m) + (\varphi(\pi(m))D(a))(x),$$

we have

$$(\varphi(a)f_v - f_v\cdot a)(x) = f_v(\varphi(a)x - a\cdot x)$$

$$= M_v(\mu_{a\cdot x-\varphi(a)x}) = lim_\alpha m_v^\alpha(\mu_{a\cdot x-\varphi(a)x})$$

$$= M_v(\varphi(a)\mu_x - a\cdot\mu_x) + lim_\alpha(\varphi(\pi(m_v^\alpha))D(a))(x)$$

$$= (\varphi(a)M_v - M_v\cdot a)(\mu_x) + (\pi''(M_v)(\varphi)D(a))(x).$$

Thus

$$\|(\varphi(a)f_v - f_v\cdot a)(x) - D(a)(x)\| = \|(\varphi(a)M_v - M_v\cdot a)(\mu_x) + (\pi''(M_v)(\varphi)D(a))(x) - D($$

$$\leq \|(\varphi(a)M_v - M_v\cdot a)(\mu_x)\| + \|(\pi''(M_v)(\varphi)D(a))(x) - D(a)(x)\|$$

$$\leq \|\varphi(a)M_v - M_v\cdot a\|\|\mu_x\| + \|\pi''(M_v)(\varphi) - 1\|\cdot\|x\|\cdot\|D(a)\|$$

$$= \|\varphi(a)M_v - M_v\cdot a\|\|D\|\|x\| + \|\pi''(M_v)(\varphi) - 1\|\cdot\|x\|\cdot\|D(a)\|.$$

Thus $D(a) = lim_v(\varphi(a)f_v - f_v\cdot a) \quad (a\in A)$. It follows that A is approximately left φ-amenable. Since (ii) clearly implies (i) the equivalences follow. The right version can be prove similarly. □

Proposition 3.1.6. *Let A and B be Banach algebras and $\varphi\in\Phi_B$. Suppose $\tau : A\to B$ is a continuous epimorphism. Then if A is approximately left [right] $\varphi\circ\tau$-amenable, then B is approximately left [right] φ-amenable.*

Proof. Let $X\in\mathcal{M}^A_{\varphi_r}$ and $Y = X$ be the A-bimodule with module actions induced via τ as follows; the left module action defined by $a\cdot x = \tau(a)x$ and right module action defined by $x\cdot a = (\varphi\circ\tau)(a)x \quad (a\in A, x\in X)$. If $D : B\to X'$ is a derivation, then $D\circ\tau : A\to Y'$ is a derivation. Thus if A is approximately left $\varphi\circ\tau$- amenable, there is a net (λ_v) in X' with

$$(D\circ\tau)(a) = lim_v[(\varphi\circ\tau)(a)\lambda_v - lambda_v\cdot h(a)].$$

Thus D is approximately inner and so B is approximately φ- amenable. An analogous argument works on the right. □

Remark 3.1.7. This argument does not extend to the closure of a homomorphic image, since the net (λ_v) in X' need not be bounded.

Corollary 3.1.8. *Let A and B be Banach algebras and $\varphi \in \Phi_B$. Suppose $\tau : A \to B$ is a continuous epimorphism. Then if A is approximately left [right] character amenable, then B is approximately left [right] character-amenable.*

Proof. This follows from Proposition 3.6. □

Corollary 3.1.9. *Let A be a Banach algebra. Suppose A is approximately left [right] character amenable, and I is a closed two-sided ideal of A. Then A/I is approximately left [right] character amenable. If I is left [right] character amenable and A/I is approximately left [right] character amenable, then A is left [right] approximately character amenable.*

Proof. Immediate from Proposition 3.6. In the reverse direction, the standard argument [55, Theorem 2.6 (iii)] applies. □

Proposition 3.1.10. *Let A be a Banach algebra without identity and A_e denote the Banach algebra obtained by adjoining an identity e. Let $\varphi \in \Phi_A$ and let φ_e be the unique extension of φ to an element of Φ_{A_e}. Then A is approximately left [right] φ-amenable if and only if A_e is approximately left [right] φ_e-amenable. In particular, the unitization algebra A_e is approximately left [right] character amenable if and only if A is approximately left [right] character amenable.*

Proof. This can be shown by simple modifications of the argument used in the proof of [35, Proposition]. □

Given two Banach algebras A and B, for $f \in A'$ and $g \in B'$, let $f \otimes g$ denote the element of $(A\hat{\otimes}B)'$ satisfying $(f \otimes g)(a \otimes b) = f(a)g(b)$. With this, we have that $\Phi_{A\hat{\otimes}B} = \{\varphi\otimes\psi : \varphi\in\Phi_A, \psi\in\Phi_B\}$.

The following result is from [**?**]:

Theorem 3.1.11. *Let A and B be Banach algebras and let $\varphi\in\Phi_A$ and $\psi\in\Phi_B$. Then $A\hat{\otimes}B$ is $(\varphi\otimes\psi)$-amenable if and only if A is φ-amenable and B is ψ-amenable.*

Remark 3.1.12. (i) We note that the concept of φ-amenable Banach algebra is the same as right φ-amenable in [55], see also [36, Theorem 1.1].

(ii) Any statement about right *varphi*-amenability turns into an analogous statement about left φ- amenability by simply replacing A by its opposite algebra.

With the above remark and Theorem 3.10, we have the following result:

Corollary 3.1.13. *Let A and B be Banach algebras. Then $A\hat{\otimes}B$ is character amenable if and only if A and B are character amenable.*

Proposition 3.1.14. *Let A and B be Banach algebras. Suppose that A is approximately character amenable and has a bounded approximate identity, and that B is character amenable. Then $A\hat{\otimes}B$ is approximately character amenable.*

Proof. The classical argument [1, Theorem 5.4] suffices to show this with due care given to the way the left and right module actions are defined. □

The next result is an analogue of a well known splitting property [35, Theorem 2.2] whose proof follows the argument of [13, Theorem 2.3]. The proof is similar.

Theorem 3.1.15. *Let A be an approximalely left character amenable Banach algebra and let X be a right Banach A-bimodule such that $x\cdot a = \varphi(a)x$ for* *some* $\varphi \in \Phi_A \quad (x \in X, a \in A)$. *Let -module*

$$\sum : 0 \to X' \overrightarrow{f} Y \overrightarrow{g} Z \to 0$$

be an admissible short exact sequence of Banach left A-modules. Then $\sum$ *approximately splits.*

Let A be a Banach algebra, we recall that a subspace X of A is called weakly complemented in A, if $X^\circ = \{\lambda \in A' : \langle \lambda, x \rangle = 0, x \in X\}$ is complemented in A'. Also, we let $A^2 = span\{ab : a, b \in A\}$ and $\overline{A^2}$ be the closure of A^2 in A.

Corollary 3.1.16. *Let* A *be an approximately left character amenable Banach algebra and let* I *be a weakly complemented left ideal of* A. *Then* I *has a right approximate identity. In particular,* $\overline{I^2} = I$.

Proof. We use the argument of [35, Corollary 2.4, Lemma 2.2] and . By Proposition 3.9, we suppose that A is unital. We consider the sequence of left A-module

$$\sum : 0 \to I \overrightarrow{i} A \to A/I \to 0,$$

and its second dual;

$$\sum'' : 0 \to I'' \overrightarrow{i''} A'' \to (I^\circ)' \to 0.$$

$\sum''$ is admissible since I is weakly complemented, and so by Theorem 3.13, there is a net of maps (I_v), each a left inverse to i'' such that $a \cdot I_v - I_v \cdot a \to 0$ for each $a \in A$. Now A'' has a right identity m, so that for $a \in I_v$,

$$(I_v \cdot a)(m) = I_v(i''\hat{a} \cdot m) = I_v(i''\hat{a}).$$

Thus

$$(a \cdot I_v)(m) = (a \cdot I_v - I_v \cdot a)(m) + \hat{a},$$

and so $a \cdot I_v \to \hat{a}, \quad a \in I$. The argument of [35, Lemma 2.2] gives the result. $\square$

Chapter 4

ARENS PRODUCTS AND AMENABILITY

4.1 Second dual of Banach algebra

Let A be a Banach algebra. Here we study the relation between approximate character amenability of A and that of its second A''.

Arens in [31] defined two products, $\Box$ and $\Diamond$, on the bidual A'' of Banach algebra A; A'' is a Banach algebra with respect to each of these products, and each algebra contains A as a closed subalgebra. The products are called the *first* and *second Arens products* on A'', respectively. For the general theory of Arens products, see [16, 20]. We recall briefly the definitions. For $\Phi \in A''$, we set

$$\langle a, \lambda \cdot \Phi \rangle = \langle \Phi, a \cdot \lambda \rangle, \quad \langle a, \Phi \cdot \lambda \rangle = \langle \Phi, \lambda \cdot a \rangle \quad (a \in A, \lambda \in A'),$$

so that $\lambda \cdot \Phi, \Phi \cdot \lambda \in A'$. Let $\Phi, \Psi \in A''$. Then

$$\langle \Phi \,\Box\, \Psi,\, \lambda\rangle = \langle \Phi,\, \Psi \,\cdot\, \lambda\rangle, \quad \langle \Phi \diamond \Psi,\, \lambda\rangle = \langle \Psi,\, \lambda \,\cdot\, \Phi\rangle \quad (\lambda \in A')\,.$$

Suppose that $\Phi, \Psi \in A''$ and that $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$ for certain nets (a_α) and (b_β) in A. Then $\Phi\Box\Psi = \lim_\alpha \lim_\beta a_\alpha b_\beta$ and $\Phi\diamond\Psi = \lim_\beta \lim_\alpha a_\alpha b_\beta$, where all limits are taken in the weak-$*$ topology $\sigma(A'', A')$ on A''.

The following result is well known:

Proposition 4.1.1. *Let A be a Banach algebra. Then both $(A'', \Box)$ and $(A'', \diamond)$ are Banach algebras containing A as a closed subalgebra.*

For a Banach algebra A, let A^{op} be the Banach algebra with the underlying Banach space as A and with product $\circ$ given by $a \circ b = ba$. A^{op} is called the opposite algebra of A.

Proposition 4.1.2. *Let A be a Banach algebra and $\varphi \in \Phi_A$. Then A is approximately left [right] φ-amenable if and only if A^{op} is approximately right [left] φ-amenable. In particular, A is approximately left [right] character amenable if and only if A^{op} is approximately right [left] character amenable.*

Proposition 4.1.3. *Let A be a Banach algebra. Suppose A is commutative. Then $(A'', \Box)$ is approximately left [right] character amenable if and only if $(A'', \diamond)$ is approximately right [left] character amenable.*

Proof. Since A is commutative, then $\lambda \cdot \Phi = \Phi \cdot \lambda \quad (\lambda \in A', \Phi \in A'')$, and $\Phi\Box\Psi = \Psi\diamond\Phi \quad (\Phi, \Psi \in A'')$, and so $(A'', \diamond) = (A'', \Box)^{op}$.

Thus the result follows by Proposition 4.2. □

Proposition 4.1.4. *Let A be a Banach algebra and suppose A admits a continuous anti-isomorphism. Then $(A'', \Box)$ is approximately left [right] character* *amenable if and only if $(A'', \Diamond)$ is approximately left [right] character amenable.*

Proof. Let $\tau : A \to A$ be continuous anti-isomorphism of A.

Let $\Phi, \Psi \in (A'', \Box)$ and let (a_α) and (b_β) be nets in A such that $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$. Let $\tau'' : (A'', \Box) \to (A'', \Diamond)$ be the second dual of τ. Then

$$\tau''(\Phi \Box \Psi) = lim_\alpha lim_\beta \tau''(a_\alpha b_\beta)$$

$$= lim_\alpha lim_\beta \tau''(b_\beta)\tau''(a_\alpha)$$

$$= \tau''(\Psi) \Diamond \tau''(\Phi).$$

Thus τ'' is an isomorphism from $(A'', \Box)$ onto $(A'', \Diamond)^{op}$ and so, the result follows from Proposition 4.2. $\Box$

The amenability, approximate amenability and character amenability versions of the result below on Banach algebra have been proved in[35], [37] and [42] respectively. The proof of character amenability and approximate amenability cases carries over. The proof is similar to the argument used in [35, Theorem 2.3].

Theorem 4.1.5. *Let A be a Banach algebra and $\varphi \in \Phi_A$. If $(A'', \Box)$ is approximately left (right) φ-amenable, then A is approximately left (right) φ-amenable. In particular, if $(A'', \Box)$ is approximately left (right) character amenable, then A is approximately left (right) character amenable.*

4.2 Algebras over locally compact groups

Let G be a locally comapct group. Then we define the group algebra $L^1(G)$ (using the left Haar measure) and the measure algebra $M(G)$ as in [16], see also [15] for details. We recall that the product $\mu, v \in M(G)$ is specified by the formula

$$\langle \mu * v, \lambda \rangle = \int_G \int_G \lambda(st) d\mu(sdv(t) \quad (\lambda \in C_0(G)),$$

so that $\delta_s * \delta_t = \delta_{st} \quad (s, t \in G)$. It is standard that $M(G)$ is a unital Banach algebra with the convolution product and it is identified with the dual space of all continuous linear functional on the Banach space $C_0(G)$ and that $L^1(G)$ is a closed ideal in $M(G)$. The subspace of $M(G)$ consisting of the continuous measures is denoted by $M_C(G)$.

The map $\varphi : \mu \to \mu(G), \quad M(G) \to \mathbf{C}$ is a character on $M(G)$, called the augumented character.

Let $C_b(G)$ be the space of bounded continuous functions on G. We recall that a function $f \in C_b(G)$ is called left uniformly continuous if the map $g \to \delta_g * f, \quad G \to C_b(G)$ is continuous (wrt the norm topology on $C_b(G)$) and $LUC(G)$ denotes the space of bounded left uniformly continuous functions on G.

The following results are from [15, Theorem 1.1] and [35, Theorem 3.1]:

Theorem 4.2.1. *Let G be a locally compact group. Then*

(i) $M(G)$ is amenable if and only if G is discrete and amenable

(ii) $M(G)$ is approximately amenable if and only if G is discrete and amenable.

It was shown in [15] that if G is non-discrete, then the complemented ideal $M_C(G)$ satisfies $\overline{M_C(G)^2} \neq M_C(G)$. With this and Corollary 3.14, we have the next result.

Theorem 4.2.2. *Let G be a locally compact group. Then $M(G)$ is approximately left character amenable if and only if G is discrete and amenable.*

Proof. Suppose G is non-discrete, then $\overline{M_C(G)^2} \neq M_C(G)$ which is impossible by Corollary 3.14. $\square$

Theorem 4.2.3. *Let G be a locally compact group. Then $(L^1(G)'', \square)$ is approximately character amenable if and only if G is finite.*

Proof. This can be shown be following similar argument in the proof of [35, Theorem 3.3] and using Corollary 3.8, Theorem 4.2 and Corollary 3.14. $\square$

Corollary 4.2.4. *Let G be a locally compact group. Then $LUC(G)'$ is approximately character amenable if and only if G is finite.*

Chapter 5

THEORY OF *NA(H)*-CLASSES

5.1 Introduction

In this chapter we study normal and norm-attainable operators in *NA(H)*-Classes. We concentrate on some of their properties, for example, normality and denseness. We also consider conditions under which some elementary operators become norm-attainable. Lastly, we take a keener look on amenability of *NA(H)*-Classes.

5.2 Norm-attainable operators

Norm-attainable operators have many applications, some of which are discussed in [21]. Let $\mathcal{NA}(H)$ denote the algebra of all norm-attainable operators on H. The denseness of $A \in \mathcal{NA}(H)$ was established in [21] and, in a more constructive way in [22]. We begin this section with our own constructive proof of the denseness of A. This proof may not be new but we were unable to find it in the literature and we need the specific details of this proof for our study of denseness for norm-attainable operator-valued functions.

Lemma 5.2.1. *Let B be a non-negative bounded linear operator on H. For every $\epsilon > 0$ there exists a rank-1 non-negative operator, C, such that $\|C\| \leq \epsilon$ and $(B+C)x = \|B+C\|x$ for some unit vector $x \in (Ker(B))^{\perp}$.*

Proof. Let $\epsilon > 0$. Since

$$\|B\| = \sup_{\|x\|=1} \langle Bx, x\rangle = \sup\{\langle Bx, x\rangle : \|x\| = 1,\ x \in (Ker(B))^{\perp}\},$$

there exists a unit vector $y \in (Ker(B))^{\perp}$ such that $\|B\| - \frac{\epsilon}{2} \leq \langle By, y\rangle$. Define $C = \epsilon\langle ., y\rangle y$. Then C is non-negative a rank-1 operator of norm ϵ. Moreover,

$$\|B+C\| = \sup_{\|x\|=1} \langle (B+C)x, x\rangle \geq \langle (B+C)y, y\rangle \geq \|B\| + \frac{\epsilon}{2}.$$

Now $B+C$ is non-negative and so $\|B+C\|$ lies in the spectrum, $\sigma(B+C)$, of $B+C$. Since C is compact, it implies $\sigma_{ess}(B+C) = \sigma_{ess}(B)$. But the spectrum of B is bounded by $\|B\|$ and hence $\|B+C\|$ must lie in the discrete spectrum of $B+C$. In other words, there exists a unit vector x such that $(B+C)x = \|B+C\|x$. Finally, we can assume without loss of generality that $x \in (Ker(B))^{\perp}$. Indeed, H decomposes as the orthogonal direct sum of $Ker(B) \oplus (Ker(B))^{\perp}$. Hence if we express $x = x^1 + x^2$ in this decomposition, we see $(B+C)x^1 = \langle x^1, y\rangle y = 0$. □

We apply the Lemma (5.2.1) for general operators. We will need to recall the properties of polar decompositions. Let $A \in L(H)$ and let $A = U|A|$ denote the polar decomposition of A. Recall that $|A| = \sqrt{A^*A}$ is a non-negative operator and U is a partial isometry. More precisely, $Ker(U) = Ker(|A|) = Ker(A)$ and U is an isometry on $(Ker(U))^{\perp}$.

Theorem 5.2.2. *Let $A \in L(H)$. There exists a sequence of rank-1 operator K_n converging to zero in operator norm such that $A + K_n$ is norm-attainable for each n. In particular, the norm-attainable operators are norm dense in $\mathcal{NA}(H)$.*

Proof. Let $A = U|A|$ be the polar decomposition of A. By Lemma 5.2.1, we know for each n there exists a non-negative rank-1 operator C_n with $\|C_n\| < 1/n$ such that for some unit vector $x_n \in (Ker(|A|))^{\perp}$, $(|A|+C_n)x_n = \||A|+C_n\|x_n$. If we premultiply this expression by U and define $K_n = UC_n$, we get $(A+K_n)x_n = \||A| + C_n\|Ux_n$.

But $x_n \in (Ker(|A|))^{\perp} = (Ker(U))^{\perp}$ U is an isometry on this set. Thus, $\|(A+K_n)x_n\| = \||A|+C_n\|$. Now, $\|(A+K_n)\| = \|U(|A|+C_n)\| \leq \|(|A|+C_n)\|$. Hence the previous equation actually shows that $\|(A + K_n)\| = \|(|A| + C_n)\|$ and this norm is attained at the vector x_n. This establishes the theorem. $\square$

This completes our constructive proof of the Lindenstrauss denseness theorem and we are ready to apply the above theorem to our study of operator-valued functions. Let $\mathbb{U}$ be an open subset of $\mathbb{C}$ and let $f : \mathbb{U} \to L(H)$ be an operator-valued function. Recall that f is continuous if for each $z \in \mathbb{U}$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $|z - w| < \delta$ implies $|f(z) - f(w)| < \epsilon$. The following theorem is analogous to Lemma 5.2.1 for continuous operator-valued functions with values in the non-negative operators.

Theorem 5.2.3. *Let $\mathbb{U}$ be an open subset of $\mathbb{C}$ and let K be a compact subset of $\mathbb{U}$. Suppose f is a continuous function on $\mathbb{U}$ with values in the non-negative linear operator on H. For every $\epsilon > 0$ there exists finite-rank non-negative* *operator P of norm ϵ such that $f(z)+P$ is norm-attainable for all $z \in K$.*

Proof. Let $\epsilon > 0$ and K be a compact subset of $\mathbb{U}$. Since f is uniformly continuous on K we can fix a $\delta > 0$ such that for every $z, w \in K$, if $|z-w| < \delta$ then $|f(z) - f(w)| < \frac{\epsilon}{8}$. Compactness implies that the collection of open balls $\{B(z,\delta)\}_{z \in K}$ admits a finite subcover $\{B(z_1,\delta), ..., B(z_n,\delta)\}$ of K. Now each $f(z)$ is a non-negative and so for each $z \in K$, there exists a unit vector ψ_z such that $\langle f(z)\psi_z, \psi_z\rangle > \|f(z)\| - \frac{\epsilon}{2}$. Let $\{\phi_1, ..., \phi_m\}$ denote the Gram- Schmidt ortho-normalization of the vectors $\{\psi_{z_1}, ..., \psi_{z_n}\}$ chosen for our finite subcover. Define $P = \epsilon \Sigma_{i=1}^{m} \langle .\phi_i\rangle \phi_i$.

Clearly, P is a non-negative finite-rank operator of norm ϵ. We claim that $f(w) + P$ is norm-attainable for each $w \in K$. Let $w \in B(z_i, \delta)$. Then

$$\begin{aligned}
\|f(w)+P\| &\geq \langle (f(w)+P)\psi_{z_i}, \psi_{z_i}\rangle \\
&= \langle (f(w)-f(z_i))\psi_{z_i}, \psi_{z_i}\rangle + \langle (f(z_i)+P)\psi_{z_i}, \psi_{z_i}\rangle \\
&> \langle (f(w)-f(z_i))\psi_{z_i}, \psi_{z_i}\rangle + \|f(z_i)\| - \frac{\epsilon}{2} + \epsilon \\
&> -\frac{\epsilon}{8} + \|f(z_i)\| + \frac{\epsilon}{2} \\
&> \|f(w)\| - \frac{\epsilon}{8} + \frac{3\epsilon}{8} \\
&> \|f(w)\|.
\end{aligned}$$

As in Lemma 5.2.1, this implies that $f(w) + P$ is norm-attainable. □

Remark 5.2.4. The previous Theorem 5.2.3 extends with exactly the same proof to hold for any bounded open domain $\mathbb{U}$.

Corollary 5.2.5. *Let $\mathbb{U}$ be a bounded open subset of $\mathbb{C}$ and suppose f is a continuous function on $\overline{\mathbb{U}}$, the closure of $\mathbb{U}$. Assume that f takes values on $\mathbb{U}$ in the non-negative linear operators on H. Then for every $\epsilon > 0$ there exists finite-*

rank non-negative operator P of norm ϵ such that $f(z)+P$ is norm-attainable for all $z \in \mathbb{U}$.

Proof. By assumption, the set $\overline{\mathbb{U}}$ is compact and hence, f is uniformly continuous on $\overline{\mathbb{U}}$. The calculation in the previous Theorem 5.2.3 goes through unchanged with the closure, $\overline{\mathbb{U}}$, of U playing the role of K.

□

5.3 Norm-attainable elementary operators

Definition 5.3.1. For an operator $S \in B(H)$ we define:

(i) Numerical range by $W(S) = \{\langle Sx, x\rangle : \ x \in H, \ \|x\| = 1\}$,

(ii) Maximal numerical range by

$$W_0(S) = \{\beta \in \mathbb{C} : \ \langle Sx_n, x_n\rangle \to \beta, \ where \ \|x_n\| = 1, \ \|Sx_n\| \to \|S\|\}.$$

Lemma 5.3.2. *Let $S \in B(H)$. δ_S is norm-attainable if there exists a vector $\zeta \in H$ such that $\|\zeta\| = 1, \ \ \|S\zeta\| = \|S\|, \ \ \langle S\zeta, \zeta\rangle = 0$.*

Proof. For any x satisfying that $x \perp \{\zeta, S\zeta\}$, define X as follows

$$X : \zeta \to \zeta, \ \ S\zeta \to -S\zeta, \ \ x \to 0,$$

because $S\zeta \perp \zeta$. Since X is a bounded operator on H and $\|X\zeta\| = \|X\| = 1$,

$$\|SX\zeta - XS\zeta\| = \|S\zeta - (-S\zeta)\| = 2\|S\zeta\| = 2\|S\|.$$

It follows that $\|\delta_S\| = 2\|S\|$ via the result in [78], because $\langle S\zeta, \zeta\rangle = 0 \in W_0(S)$. Hence we have that $\|SX - XS\| = 2\|S\| = \|\delta_S\|$. Therefore, δ_S is norm-attainable. □

Theorem 5.3.3. *Let $S \in B(H)$, $\beta \in W_0(S)$ and $\alpha > 0$. There exists an operator $Z \in B(H)$ such that $\|S\| = \|Z\|$, with $\|S - Z\| < \alpha$. Furthermore, there exists a vector $\eta \in H$, $\|\eta\| = 1$ such that $\|Z\eta\| = \|Z\|$ with $\langle Z\eta, \eta\rangle = \beta$.*

Proof. Without loss of generality, we may assume that $\|S\| = 1$. Let $x_n \in H$ $(n = 1, 2, ...)$ be such that $\|x_n\| = 1$, $\|Sx_n\| \to 1$ and

$$\lim_{n\to\infty} \langle Sx_n, x_n\rangle = \beta.$$

Consider a partial isometry G and $L = \int_0^1 \beta dE_\beta$, the spectral decomposition of L. Let $S = GL$, the polar decomposition of S. Since

$$\lim_{n\to\infty} \|Sx_n\| = \|S\| = \|L\| = 1,$$

we have that $\|Lx_n\| \to 1$ as n tends to ∞ and

$$\lim_{n\to\infty} \langle Sx_n, x_n\rangle = \lim_{n\to\infty} \langle GLx_n, x_n\rangle = \lim_{n\to\infty} \langle Lx_n, G^* x_n\rangle.$$

Now for $H = \overline{Ran(L)} \oplus KerL$, we can choose x_n such that $x_n \in \overline{Ran(L)}$ for large n. Indeed, let

$$x_n = x_n^{(1)} \oplus x_n^{(2)},\ n = 1, 2, ...$$

Then we have that

$$Lx_n = Lx_n^{(1)} \oplus Lx_n^{(2)} = Lx_n^{(1)}$$

and that

$$\lim_{n\to\infty} \|x_n^{(1)}\| = 1,\ \lim_{n\to\infty} \|x_n^{(2)}\| = 0$$

since

$$\lim_{n\to\infty} \|Lx_n\| = 1.$$

Replacing x_n with $\frac{x_n^{(1)}}{\|x_n^{(1)}\|}$, we obtain

$$\lim_{n\to\infty} \left\| L\frac{1}{\|x_n^{(1)}\|} x_n^{(1)} \right\| = \lim_{n\to\infty} \left\| S\frac{1}{\|x_n^{(1)}\|} x_n^{(1)} \right\| = 1,$$

$$\lim_{n\to\infty}\left(S\frac{1}{\|x_n^{(1)}\|}x_n^{(1)}, \frac{1}{\|x_n^{(1)}\|}x_n^{(1)}\right) = \beta.$$

Now assume that $x_n \in \overline{RanL}$. Since G is a partial isometry from $\overline{RanL}$ onto $\overline{RanS}$, we have that $\|Gx_n\| = 1$ and $\lim_{n\to\infty}\langle Lx_n, G^* x_n\rangle = \beta$. For L is a positive operator, $\|L\| = 1$ and for any $x \in H$,

$$\langle Lx, x\rangle \leq \langle x, x\rangle = \|x\|^2.$$

Replacing x with $L^{\frac{1}{2}}x$, we get that $\langle L^2 x, x\rangle \leq \langle Lx, x\rangle$, where $L^{\frac{1}{2}}$ is the positive square root of L. Therefore we have that $\|Lx\|^2 = \langle Lx, Lx\rangle \leq \langle Lx, x\rangle$. It is obvious that $\lim_{n\to\infty}\|Lx_n\| = 1$ and that

$$\|Lx_n\|^2 \leq \langle Lx_n, x_n\rangle \leq \|Lx_n\|^2 = 1.$$

Hence, $\lim_{n\to\infty}\langle Lx_n, x_n\rangle = 1 = \|L\|$. Moreover, Since $I - L \geq 0$, we have $\lim_{n\to\infty}\langle (I-L)x_n, x_n\rangle = 0$. thus $\lim_{n\to\infty}\|(I-L)^{\frac{1}{2}}x_n\| = 0$. Indeed,

$$\lim_{n\to\infty}\|(I-L)x_n\| \leq \lim_{n\to\infty}\|(I-L)^{\frac{1}{2}}\|.\|(I-L)^{\frac{1}{2}}x_n\| = 0.$$

For $\alpha > 0$, let $\gamma = [0, 1-\frac{\alpha}{2}]$ and let $\rho = [1-\frac{\alpha}{2}, 1]$. We have

$$\begin{aligned} L &= \int_\gamma \mu dE_\mu + \int_\rho \mu dE_\mu \\ &= \int_0^{1-\frac{\alpha}{2}} \mu dE_\mu + \int_{1-\frac{\alpha}{2}+0}^{1} \mu dE_\mu \\ &= LE(\gamma) \oplus LE(\rho). \end{aligned}$$

Next we show that $\lim_{n\to\infty}\|E(\gamma)x_n\| = 0$. If there exists a subsequence x_{n_i}, $(i = 1, 2, ...,)$ such that $\|E(\gamma)x_{n_i}\| \geq \epsilon > 0$, $(i = 1, 2, ...,)$, then since $\lim_{i\to\infty}\|x_{n_i} - Lx_{n_i}\| = \lim_{i\to\infty}(I-L)x_{n_i} = 0$, it follows that

$$\begin{aligned} \lim_{i\to\infty}\|x_{n_i} - Lx_{n_i}\| &= \lim_{i\to\infty}(\|E(\gamma)x_{n_i} - LE(\gamma)x_{n_i}\|^2 + \|E(\rho)x_{n_i} - LE(\rho)x_{n_i}\|^2) \\ &= 0. \end{aligned}$$

Hence we have that $\lim_{i\to\infty}(\|E\gamma)x_{n_i} - LE(\gamma)x_{n_i}\|^2 = 0$. Now it is clear that

$$\begin{aligned}
\|E\gamma)x_{n_i} - LE(\gamma)x_{n_i}\| &\geq \|E\gamma)x_{n_i}\| - \|LE(\gamma)\|.\|E\gamma)x_{n_i}\| \\
&\geq (I - \|LE(\gamma)\|).\|E\gamma)x_{n_i}\| \\
&\geq \frac{\alpha}{2}\epsilon \\
&> 0.
\end{aligned}$$

This is a contradiction. Therefore,

$$\lim_{n\to\infty} \|E(\gamma)x_n\| = 0.$$

Since

$$\lim_{n\to\infty} \langle Lx_n, x_n\rangle = 1,$$

we have that

$$\lim_{n\to\infty} \langle LE(\rho)x_n, E(\rho)x_n\rangle = 1$$

and

$$\lim_{n\to\infty} \langle E(\rho)x_n, G^* E(\rho)x_n\rangle = \beta.$$

It is easy to see that

$$\lim_{n\to\infty} \|E(\rho)x_n\| = 1, \ \lim_{n\to\infty} \left(L\frac{E(\rho)x_n}{\|E(\rho)x_n\|}, \frac{E(\rho)x_n}{\|E(\rho)x_n\|}\right) = 1$$

and

$$\lim_{n\to\infty} \left(L\frac{E(\rho)x_n}{\|E(\rho)x_n\|}, G^*\frac{E(\rho)x_n}{\|E(\rho)x_n\|}\right) = \beta$$

Replacing x with $\frac{E(\rho)x_n}{\|E(\rho)x_n\|}$, we can assume that $x_n \in E(\rho)H$ for each n and $\|x_n\| = 1$. Let

$$\begin{aligned}
J &= \int_\gamma \mu dE_\mu + \int_\rho \mu dE_\mu \\
&= \int_0^{1-\frac{\alpha}{2}} \mu dE_\mu + \int_{1-\frac{\alpha}{2}+0}^{1} \mu dE_\mu \\
&= J_1 \oplus E(\rho).
\end{aligned}$$

Then it is evident that

$$\|J\| = \|S\| = \|L\| = 1, Jx_n = x_n,$$

and $\|J - L\| < \frac{\alpha}{2}$. If we can find a contraction V such that $V - G < \frac{\alpha}{2}$ and $\|Vx_n\| = 1$ for a large n, $\langle Vx_n, x_n\rangle = \beta$, then letting $Z = VJ$, we have that $\|Zx_n\| = \|VJx_n\| = 1$, and that

$$\langle Zx_n, x_n\rangle = \langle VJx_n, x_n\rangle = \langle Vx_n, x_n\rangle = \beta,$$

$$\begin{aligned}
\|S - Z\| &= \|GL - VJ\| \\
&\leq \|GL - GJ\| + \|GJ - VJ\| \\
&\leq \|G\|.\|L - J\| + \|G - V\|.\|J\| \\
&\leq \frac{\alpha}{2} + \frac{\alpha}{2} \\
&= \alpha.
\end{aligned}$$

To finish the proof, we now construct the desired contraction V. Clearly, $\lim_{n\to\infty}\langle x_n, G^* x_n\rangle = \beta$, because $\lim_{n\to\infty}\langle Lx_n, G^* x_n\rangle = \beta$ and

$$\lim_{n\to\infty} \|x_n - Lx_n\| = 0.$$

Let $Gx_n = \phi_n x_n + \varphi_n y_n, \ \ (y_n \perp x_n, \ \|y_n\| = 1)$ then $\lim_{n\to\infty} \phi_n = \beta$, because

$$\lim_{n\to\infty} \langle Gx_n, x_n\rangle = \lim_{n\to\infty} \langle x_n, G^* x_n\rangle = \beta$$

but $\|Gx_n\|^2 = |\phi_n|^2 + |\varphi_n|^2 = 1$, so we have that $\lim_{n\to\infty} |\varphi_n| = \sqrt{1 - |\beta|^2}$. Now for $\alpha > 0$, there exists an integer M such that $|\phi_M - \beta| < \frac{\alpha}{8}$. Choose φ_M^0 such that $|\varphi_M^0| = \sqrt{1 - |\beta|^2}, \ \ |\varphi_M - \varphi_M^0| < \frac{\alpha}{8}$. We have that

$$\begin{aligned}
Gx_M &= \phi_M x_M + \varphi_M y_M - \beta x_M + \beta x_M - \varphi_M^0 y_M + \varphi_M^0 y_M \\
&= (\phi - \beta) x_M + (\varphi_M - \varphi_M^0) y_M + \beta x_M + \varphi_M^0 y_M.
\end{aligned}$$

Let $q_M = \beta x_M + \varphi_M^0 y_M$,

$$Gx_M = (\phi - \beta)x_M + (\varphi_M - \varphi_M^0)y_M + q_M.$$

Suppose that $y \perp x_M$, then

$$\begin{aligned}\langle Gx_M, Gy\rangle &= (\phi - \beta)\langle x_M, Gy\rangle + (\varphi_M - \varphi_M^0)\langle y_M, Gy\rangle + \langle q_M, Gy\rangle \\ &= 0,\end{aligned}$$

because G^*G is a projection from H to $RanL$. It follows that

$$|\langle q_M, Gy\rangle| \leq |\phi_M - \beta|.\|y\| + |\varphi_M - \varphi_M^0|.\|y\| \leq \frac{\alpha}{4}\|y\|.$$

If we suppose that $Gy = \phi q_M + y^0$, $(y^0 \perp q_M,)$ then y^0 is uniquely determined by y. Hence we can define V as follows

$$V : x_M \to q_M,\ y \to y^0,\ \phi x_M + \varphi_M y \to \phi q_M + \varphi_M y^0,$$

with both ϕ, φ being complex numbers. V is a linear operator. We prove that V is a contraction. Now,

$$\|Vx_M\|^2 = \|q_M\|^2 = |\beta|^2 = |\varphi_M^0|^2 = 1,$$

$$\|Vy\|^2 = \|Gy\|^2 - |\phi y|^2 \leq \|Gy\|^2 \leq \|y\|^2.$$

It follows that

$$\|V\phi\|^2 = \|\phi\|^2\|Vx_M\|^2 + |\varphi|^2\|Vy\|^2 \leq |\phi|^2 + |\varphi|^2 = 1,$$

for each $x \in H$ satisfying that $x = \phi x_M + \varphi_M y,\quad \|x\| = 1,\ x_M \perp y$, which is equivalent to that V is a contraction. From the definition of V, we can show that

$$\|Gx_M - Vx_M\|^2 = |\phi - \beta|^2 + |\varphi_M - \varphi_M^0|^2 \leq \frac{2\alpha^2}{16} = \frac{1}{8}\alpha^2.$$

If $y \perp x_M,\ \|y\| \leq 1$ then obtain

$$\|Gy - Vy\| = |\phi|\|Vx_M\| = |\langle Gy, Vx_M\rangle| = |\langle q_M, Gy\rangle| < \frac{\alpha}{4}.$$

Hence for any $x \in H,\ x = \phi x_M + \varphi_M y,\ \ \|x\| = 1,$

$$\begin{aligned}
\|Gx - Vx\|^2 &= \|\phi(G-V)x_M + \varphi(G-V)y\|^2 \\
&= |\phi|^2\|(G-V)x_M\|^2 + |\varphi|^2\|(G-V)y\|^2 \\
&< |\phi|^2.\frac{\alpha^2}{16} + |\varphi|^2.\frac{\alpha^2}{16} \\
&< \frac{\alpha^2}{8},
\end{aligned}$$

which implies that

$$\|(G-V)x\| < \frac{\alpha}{2},\ \ \|x\| = 1,$$

and hence $\|(G-V)\| < \frac{\alpha}{2}$. Let $S = VJ$. Then S is what we want and this completes the proof. □

Theorem 5.3.4. *The set of operators $S \in B(H)$ implementing a norm-attainable inner derivation, δ_S^N, is uniformly dense in $B(H)$.*

Proof. Let $\alpha > 0$ and $S \in B(H)$. It is clear that $\delta_S^N = \delta_{S-\beta}^N$ for any $\beta \in \mathbb{C}$. From [**??**], there is a $\beta \in \mathbb{C}$ such that $0 \in W_0(S-\beta)$ and

$$\|\delta_S^N\| = \|\delta_{S-\beta}^N\| = 2\|S-\beta\|.$$

From Theorem 5.3.3, there exists an operator $Z \in B(H)$ such that

$$\|S-\beta\| = \|Z\|,\ \ \|S-\beta-Z\| < \alpha$$

and there exists a vector $\eta \in H$ such that $\|Z\eta\| = \|Z\|,\ \ \langle Z\eta, \eta\rangle = 0$. By Lemma 5.3.2, δ_Z^N is norm-attainable which is equivalent to that $\delta_{Z+\beta}^N$ is norm-attainable and implies that

$$\|S-(Z+\beta)\| = \|S-\beta-Z\| = 2\|S-\beta\| < \alpha.$$

$\square$

Next we consider generalized derivations.

Theorem 5.3.5. *Let $S, T \in B(H)$ If there exists vectors $\zeta, \eta \in H$ such that $\|\zeta\| = \|\eta\| = 1$, $\|S\zeta\| = \|S\|$, $\|T\eta\| = \|T\|$ and $\frac{1}{\|S\|}\langle S\zeta, \zeta\rangle = -\frac{1}{\|T\|}\langle T\eta, \eta\rangle$, then $\delta^N_{S,T}$ is norm-attainable.*

Proof. By linear dependence of vectors, if η and $T\eta$ are linearly dependent, i.e.,$T\eta = \phi\|T\|\eta$, then it is true that $|\phi| = 1$ and $|\langle T\eta, \eta\rangle| = \|T\|$. It follows that $|\langle S\zeta, \zeta\rangle| = \|S\|$ which implies that $S\zeta = \varphi\|S\|\zeta$ and $|\varphi| = 1$. Hence $\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle = \varphi = -\left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle = -\phi$. Defining X as $X : \eta \to \zeta,\ \ \{\zeta\}^{\perp} \to 0$, we have $\|X\| = 1$ and $(SX - XT)\eta = \varphi(\|S\| + \|T\|)\zeta$, which implies that

$$\|SX - XT\| = \|(SX - XT)\eta\| = \|S\| + \|T\|.$$

It follows that

$$\|SX - XT\| = \|S\| + \|T\| = \|\delta^N_{S,T}\|.$$

That is $\delta^N_{S,T}$ is norm-attainable. If η and $T\eta$ are linearly independent, then $\left|\left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle\right| < 1$, which implies that $\left|\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle\right| < 1$. Hence ζ and $S\zeta$ are also linearly independent. Let us redefine X as follows: $X : \eta \to \zeta,\ \ \frac{T\eta}{\|T\|} \to -\frac{S\zeta}{\|S\|},\ \ x \to 0$, where $x \in \{\eta, T\eta\}^{\perp}$. We show that X is a partial isometry. Let

$$\frac{T\eta}{\|T\|} = \left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle \eta + \tau h, \ \ \|h\| = 1,\ h \perp \eta.$$

Since η and $T\eta$ are linearly independent, $\tau \neq 0$. So we have that

$$X\frac{T\eta}{\|T\|} = \left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle X\eta + \tau Xh = -\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle \zeta + \tau Xh,$$

which implies that $\left\langle X\frac{T\eta}{\|T\|}, \zeta \right\rangle = -\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle + \tau\langle Xh, \zeta\rangle = -\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle$.

It follows then that $\langle Xh, \zeta\rangle = 0$ i.e., $Xh \perp \zeta (\zeta = X\eta)$. Hence we have that

$$\left\|\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle \zeta\right\|^2 + \|\tau Xh\|^2 = \left\|X\frac{T\eta}{\|T\|}\right\|^2 = \left|\left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle\right|^2 + |\tau|^2 = 1,$$

which implies that $\|Xh\| = 1$. Now it is evident that X a partial isometry and $\|(SX - XT)\zeta\| = \|SX - XT\| = \|S\| + \|T\|$, which is equivalent to $\|\delta^N_{S,T}(X)\| = \|S\| + \|T\|$. So, $\|\delta^N_{S,T}\| = \|S\| + \|T\|$. Hence, $\delta^N_{S,T}$ is norm-attainable. $\square$

Lemma 5.3.6. *The set of operators $S, T \in B(H)$ which are implementing a norm-attainable generalized derivation, $\delta^N_{S,T}$, is uniformly dense in $B(H) \times B(H)$.*

Proof. Suppose either S or T is a scalar operator for example $T = \kappa I$, for some scalar κ, then $\delta^N_{S,T}(X) = SX - X\kappa I$. Clearly, $\|\delta^N_{S,T}\| = \|S - \kappa\|$. From Theorem 5.3.3, for $\alpha > 0$, there exists an operator $C \in B(H)$ and a vector $\zeta \in H$ such that $\|C\zeta\| = \|C\|$ and $\|S - \kappa - C\| < \alpha$. Let $Z = C + \kappa I,\ \ T_0 = \kappa I$. Therefore, we have that $\delta^N_{Z,T_0}(X) = CX, \quad \forall\ X \in B(H)$. Let P be the projection on $\{\zeta\}$. Then $\|\delta^N_{Z,T_0}(P)\| = \|P\|$, which implies that δ^N_{Z,T_0} is norm-attainable and $\|\langle S, T\rangle - \langle Z - T_0\rangle\| < \alpha$. Suppose neither S nor T is a scalar operator, then there is a complex number β such that

$$\|\delta^N_{S,T}\| = \|S - \beta\| + \|T - \beta\|$$

and $W_0\left(\frac{S-\beta}{\|S-\beta\|}\right) \cap W_0\left(\frac{T-\beta}{\|T-\beta\|}\right)$ is nonempty. Let $\nu \in W_0\left(\frac{S-\beta}{\|S-\beta\|}\right) \cap W_0\left(\frac{T-\beta}{\|T-\beta\|}\right)$. Then for $\alpha > 0$, from Theorem 5.3.3, there exists Z and T_0 and $\zeta, \eta \in H$ such that $\|Z\| = \|T_0\| = 1, \quad \|\zeta\| = \|\eta\| = 1, \quad \left\|\frac{S-\beta}{\|S-\beta\|} - Z\right\| < \frac{\alpha}{2}(\|\delta^N_{S,T}\| + 1)$ and $\left\|\frac{T-\beta}{\|T-\beta\|} - T_0\right\| < \frac{\alpha}{2}(\|\delta^N_{S,T}\| + 1)$, with

$$\|Z\zeta\| = \|Z\|, \ \ \|T_0\eta\| = \|T_0\|, \ \ \langle Z\zeta, \zeta\rangle = \nu, \ \ \langle T_0\eta, \eta\rangle = -\nu.$$

Let $Z_0 = \|S - \beta\|Z,\ \ T_{00} = \|T - \beta\|T_0$. Then $\left\langle \frac{Z_0\zeta}{\|Z_0\|}, \zeta\right\rangle = \nu = -\left\langle \frac{T_{00}\eta}{\|T_{00}\|}, \eta\right\rangle$. By Theorem 5.3.5, $\delta^N_{Z_0,T_{00}}$ is norm-attainable. But we have that $\|S - \beta - Z_0\| < \frac{\alpha}{2}$ and $\|T - \beta - T_{00}\| < \frac{\alpha}{2}$ hence $\|\langle S, T\rangle - \langle Z_0 + \beta, T_{00} + \beta\rangle\| < \alpha$, and $\delta^N_{\langle Z_0+\beta, T_{00}+\beta\rangle}$ is norm-attainable.

□

Theorem 5.3.7. *Let $S, T \in B(H)$ If both S and T are norm-attainable then the basic elementary operator $M^N_{S,\,T}$ is also norm-attainable.*

Proof. For any pair (S,T) it is known that $\|M^N_{S,\,T}\| = \|S\|\|T\|$. We can assume that $\|S\| = \|T\| = 1$. If both S and T are norm-attainable, then there exists unit vectors ζ and η with $\|S\zeta\| = \|T\eta\| = 1$. We can therefore define an operator X by $X = \langle \cdot, T\eta\rangle\zeta$. Clearly, $\|X\| = 1$. Therefore, we have $\|SXT\| \geq \|SXT\eta\| = \|\|T\eta\|^2 S\zeta\| = 1$. Hence, $\|M^N_{S,\,T}(X)\| = \|SXT\| = 1$, that is $M^N_{S,\,T}$ is also norm-attainable. □

Theorem 5.3.8. *Let $S, T \in B(H)$ If both S and T are norm-attainable then the Jordan elementary operator $\mathcal{U}^N_{S,\,T}$ is also norm-attainable.*

Proof. The proof is analogous to that of the previous Lemma. □

As a remark, we give the following proposition on norm-attainable general elementary operator.

Proposition 5.3.9. *An operator $\mathcal{T}_{\tilde{S},\tilde{T}}(X) = \sum_{i=1}^{n} S_i X T_i$, is said to be norm-attainable if there is a contraction X in the unit ball, $(B(H))_1$, such that $\|\mathcal{T}_{\tilde{S},\tilde{T}}(X)\| = \|\mathcal{T}_{\tilde{S},\tilde{T}}\|$, where $\tilde{S} = (S_1, ..., S_n)$ and $\tilde{T} = (T_1, ..., T_n)$ are n-tuples in $B(H)$.*

Chapter 6

FUNCTION SPACES

6.1 Introduction

It is well known that Hankel operators constitute a very important class of operators in spaces of analytic functions. The study of these operators on different analytic spaces is not only motivated by the mathematical challenges it raises, but also by many applications on applied mathematics and in physics (see for example [96] for more information). In this chapter, we are interested on the boundedness and the compactness problem of the little Hankel operator with operator-valued symbols on weighted vector-valued Bergman spaces on the unit ball.

Throughout this paper, we fix a nonnegative integer n and let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$$

denote the $n-$dimensional Euclidean space. For

$$z = (z_1, \cdots, z_n), \qquad w = (w_1, \cdots, w_n),$$

in $\mathbb{C}^n$, we define the inner product of z and w by

$$\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n},$$

where $\overline{w_k}$ is the complex conjugate of w_k. The resulting norm is then

$$|z| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

Endowed with the above inner product, $\mathbb{C}^n$ become a Hilbert space whose canonical basis consists of the following vectors

$$e_1 = (1, 0, \cdots, 0), \quad e_2 = (0, 1, 0, \cdots, 0), \quad \cdots, \quad e_n = (0, \cdots, 0, 1).$$

The open unit ball in $\mathbb{C}^n$ is the set

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}.$$

When $\alpha > -1$, the weighted Lebesgue measure $\mathrm{d}\nu_\alpha$ in $\mathbb{B}_n$ is defined by

$$\mathrm{d}\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha \mathrm{d}\nu(z), \qquad z \in \mathbb{B}_n$$

where $\mathrm{d}\nu$ is the Lebesgue measure in $\mathbb{C}^n$ and

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)}$$

is the normalizing constant so that $\mathrm{d}\nu_\alpha$ becomes a probability measure on $\mathbb{B}_n$. A function defined on the unit ball $\mathbb{B}_n$ will be called a vector-valued function when it takes its values in some vector space. If X is a complex Banach space, a vector-valued function $f : \mathbb{B}_n \longrightarrow X$ (a X-valued function) is said to be strongly holomorphic in $\mathbb{B}_n$ if for every $z \in \mathbb{B}_n$ and for every $k \in \{1, \cdots, n\}$, the limit

$$\lim_{\lambda \longrightarrow 0} \frac{f(z + \lambda e_k) - f(z)}{\lambda}$$

exists in X, where $\lambda \in \mathbb{C} - \{0\}$. The space of all $X-$valued strongly holomorphic functions on $\mathbb{B}_n$ will be denoted by $\mathcal{H}(\mathbb{B}_n, X)$. We will also denote by $H^\infty(\mathbb{B}_n, X)$

the space of all bounded X-valued holomorphic functions. Let $X^\star$ denotes the space of all bounded linear functionals $x^\star : X \longrightarrow \mathbb{C}$ (the topological dual space of X). We say that a vector-valued function $f : \mathbb{B}_n \longrightarrow X$ is weakly holomorphic if for every $x^\star \in X^\star$, the scalar-valued function $x^\star(f) : \mathbb{B}_n \longrightarrow \mathbb{C}$ is holomorphic in the usual sense. An important result by N. Dunford ([90]) shows that a vector-valued function is strongly holomorphic if and only if it is weakly holomorphic.

6.1.1 The conjugate $\overline{X}$ of the complex Banach space X

We will use the following definition and notation which can be found in [94]. Let $x \in X$, $x^\star \in X^\star$ and $\lambda \in \mathbb{C}$. We have the following identities

$$\langle \lambda x, x^\star \rangle_{X,X^\star} = \lambda \langle x, x^\star \rangle_{X,X^\star} = \langle x, \overline{\lambda} x^\star \rangle_{X,X^\star},$$

so that we have a regular rule of an inner product. The complex conjugate $\overline{x}$ of $x \in X$, is the linear functional on $X^\star$ defined by

$$\overline{x}(x^\star) = \overline{\langle x, x^\star \rangle}_{X,X^\star},$$

for every $x^\star \in X^\star$. Therefore,

$$\overline{X} = \{\overline{x} : x \in X\}$$

is called the complex conjugate of the Banach space X. With the norm defined by

$$\|\overline{x}\| := \sup_{\|x^\star\|_{X^\star}=1} |\overline{x}(x^\star)|,$$

$\overline{X}$ becomes a Banach space. Moreover, we have that $\|x\|_X = \|\overline{x}\|_{\overline{X}}$ for any $x \in X$, so that X and $\overline{X}$ are isometrically anti-isomorphic.

6.1.2 Vector-valued spaces

In the sequel, we will integrate vector-valued measurable functions in the sense of Bochner (see [90] for more information). Let X be a complex Banach space. A measurable function $f : \mathbb{B}_n \longrightarrow X$ is Bochner-integrable with respect to the measure ν_α in the unit ball $\mathbb{B}_n$ if and only if the Lebesgue integral

$$\|f\|_{1,\alpha,X} = \int_{\mathbb{B}_n} \|f(z)\|_X \mathrm{d}\nu_\alpha(z)$$

is finite. For $0 < p < \infty$, the Bochner-Lebesgue space $L^p_{\nu_\alpha}(\mathbb{B}_n, X)$ consists of all vector-valued measurable functions $f : \mathbb{B}_n \longrightarrow X$ such that

$$\|f\|^p_{p,\alpha,X} = \int_{\mathbb{B}_n} \|f(z)\|^p_X \mathrm{d}\nu_\alpha(z) < \infty.$$

The vector-valued Bergman space $A^p_\alpha(\mathbb{B}_n, X)$ is defined by

$$A^p_\alpha(\mathbb{B}_n, X) = L^p_{\nu_\alpha}(\mathbb{B}_n, X) \cap \mathcal{H}(\mathbb{B}_n, X).$$

The weak Bochner-Lebesgue space $L^{p,\infty}_\alpha(\mathbb{B}_n, X)$ consists of all vector-valued measurable functions $f : \mathbb{B}_n \longrightarrow X$ for which

$$\|f\|_{L^{p,\infty}_\alpha(\mathbb{B}_n,X)} = \left(\sup_{\lambda>0} \lambda^p \nu_\alpha\left(\{z \in \mathbb{B}_n : \|f(z)\|_X > \lambda\}\right)\right)^{1/p} < \infty.$$

The weak vector-valued Bergman space $A^{p,\infty}_\alpha(\mathbb{B}_n, X)$ is defined by

$$A^{p,\infty}_\alpha(\mathbb{B}_n, X) = \mathcal{H}(\mathbb{B}_n, X) \cap L^{p,\infty}_\alpha(\mathbb{B}_n, X).$$

Let X, Y be two complex Banach spaces and $\alpha > -1$. We have the following two lemmas whose proofs can be found in [94].

Lemma 6.1.1. *Let $T : X \longrightarrow Y$ be a bounded linear operator. If $f : \mathbb{B}_n \longrightarrow X$ is $\nu_\alpha-$Bochner integrable in the unit ball, then $Tf : \mathbb{B}_n \longrightarrow Y$ is $\nu_\alpha-$Bochner integrable in the unit ball and we have*

$$\int_{\mathbb{B}_n} Tf(z)\mathrm{d}\nu_\alpha(z) = T\left(\int_{\mathbb{B}_n} f(z)\mathrm{d}\nu_\alpha(z)\right).$$

Lemma 6.1.2. *If $f : \mathbb{B}_n \longrightarrow X$ is a ν_α-Bochner integrable vector-valued function in the unit ball, then the following inequality holds*

$$\left\| \int_{\mathbb{B}_n} f(z) \mathrm{d}\nu_\alpha(z) \right\|_X \leq \int_{\mathbb{B}_n} \|f(z)\|_X \mathrm{d}\nu_\alpha(z).$$

6.1.3 Vector-valued Lipschitz spaces and vector-valued γ-Bloch spaces

The radial derivative of a vector-valued holomorphic function $f : \mathbb{B}_n \longrightarrow X$ denoted Nf is defined for $z \in \mathbb{B}_n$ by

$$Nf(z) := \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z). \tag{6.1.3.1}$$

Let $f \in \mathcal{H}(\mathbb{B}_n, X)$ and

$$f(z) = \sum_{k=0}^{\infty} f_k(z), \qquad z \in \mathbb{B}_n$$

the homogeneous expansion of the function f where f_k are homogeneous holomorphic polynomials of degree k with coefficients in X. For any two real parameters α and t such that neither $n+\alpha$ nor $n+\alpha+t$ is a negative integer, we define an invertible operator $R^{\alpha,t} : \mathcal{H}(\mathbb{B}_n, X) \to \mathcal{H}(\mathbb{B}_n, X)$ as

$$R^{\alpha,t} f(z) := \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha)\Gamma(n+1+k+\alpha+t)}{\Gamma(n+1+\alpha+t)\Gamma(n+1+k+\alpha)} f_k(z), \tag{6.1.3.2}$$

where $z \in \mathbb{B}_n$ and Γ is the classical Euler Gamma function. For $\gamma \geq 0$, we denote by $\Gamma_\gamma(\mathbb{B}_n, X)$ the space of vector-valued holomorphic functions $f : \mathbb{B}_n \longrightarrow X$ for which there exists an integer $k > \gamma$ such that

$$\|f\|_{\gamma,X} = \|f(0)\|_X + \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|N^k f(z)\|_X < \infty,$$

where $N^k = N \circ N \circ \cdots \circ N$ $k-$times. The definition of the space $\Gamma_\gamma(\mathbb{B}_n, X)$ is independent of the integer k used. The space $\Gamma_\gamma(\mathbb{B}_n, X)$ will be called the vector-valued holomorphic Lipschitz space and for $\gamma = 0$, we write $\mathcal{B}(\mathbb{B}_n, X) = \Gamma_0(\mathbb{B}_n, X)$. It is clear that $f \in \mathcal{B}(\mathbb{B}_n, X)$ if and only if f is a vector-valued holomorphic function and

$$\|f\|_{\mathcal{B}(\mathbb{B}_n,X)} = \|f(0)\|_X + \sup_{z\in\mathbb{B}_n} (1 - |z|^2)\|Nf(z)\|_X < \infty.$$

That is, $\mathcal{B}(\mathbb{B}_n, X) = \Gamma_0(\mathbb{B}_n, X)$ is the vector-valued Bloch space. The vector-valued γ-Bloch space $\mathcal{B}_\gamma(\mathbb{B}_n, X)$ for $\gamma > 0$, is defined as the space of vector-valued holomorphic functions $f \in \mathcal{H}(\mathbb{B}_n, X)$ such that

$$\sup_{z\in\mathbb{B}_n} (1 - |z|^2)^\gamma \|Nf(z)\|_X < \infty.$$

The little vector-valued γ-Bloch space $\mathcal{B}_{\gamma,0}(\mathbb{B}_n, X)$ for $\gamma > 0$, is the subspace of $\mathcal{B}_\gamma(\mathbb{B}_n, X)$ consisting of functions f such that

$$\lim_{|z|\to 1^-} (1 - |z|^2)^\gamma \|Nf(z)\|_X = 0.$$

It is easy to see that $\mathcal{B}_1(\mathbb{B}_n, X) = \mathcal{B}(\mathbb{B}_n, X)$. Therefore, the vector-valued γ-Bloch spaces with $\gamma > 0$ generalize the vector-valued Bloch space. Let $\gamma \geq 0$. The generalized vector-valued Lipschitz space $\Lambda_\gamma(\mathbb{B}_n, X)$ consists of vector-valued holomorphic functions f in $\mathbb{B}_n$ such that for some nonnegative integer $k > \gamma$, we have

$$\|f\|_{\Lambda_\gamma(\mathbb{B}_n,X)} = \sup_{z\in\mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_X < \infty.$$

We consider the following norm on the generalized vector-valued Lipschitz space $\Lambda_\gamma(\mathbb{B}_n, X)$ by

$$\|f\|_{\Lambda_\gamma(\mathbb{B}_n,X)} = \sup_{z\in\mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_X,$$

where $k > \gamma$ is a nonnegative integer. Equipped with this norm, the generalized vector-valued Lipschitz space $\Lambda_\gamma(\mathbb{B}_n, X)$ becomes a Banach space. The generalized little vector-valued Lipschitz space $\Lambda_{\gamma,0}(\mathbb{B}_n, X)$ is the subspace of $\Lambda_\gamma(\mathbb{B}_n, X)$, which consists of functions $f \in \Lambda_\gamma(\mathbb{B}_n, X)$ such that

$$\lim_{|z|\to 1^-} (1-|z|^2)^{k-\gamma}\|R^{\alpha,k} f(z)\|_X = 0. \tag{6.1.3.3}$$

When $\gamma = 0$ and $k = 1$, then $\Lambda_0(\mathbb{B}_n, X) = \mathcal{B}(\mathbb{B}_n, X)$. It is also important to note that as in the classical case, when $0 < \gamma < 1$, we have $\Lambda_\gamma(\mathbb{B}_n, X) = \mathcal{B}_{1-\gamma}(\mathbb{B}_n, X)$.

6.1.4 Little Hankel operator with operator-valued symbol

Given two complex Banach spaces X and Y, we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators $T : X \longrightarrow Y$ endowed with the following norm

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X=1} \|Tx\|_Y = \sup_{\|x\|_X=1, \|y^\star\|_{Y^\star}=1} |\langle Tx, y^\star\rangle_{Y,Y^\star}|,$$

where $T \in \mathcal{L}(X, Y)$. Then $\mathcal{L}(X, Y)$ is a Banach space. We consider an operator-valued function $b : \mathbb{B}_n \longrightarrow \mathcal{L}(\overline{X}, Y)$ and we suppose that $b \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$. The little Hankel operator with operator-valued symbol b, denoted h_b is defined for $z \in \mathbb{B}_n$ by

$$h_b f(z) := \int_{\mathbb{B}_n} \frac{b(w)\overline{f(w)}}{(1-\langle z, w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w), \qquad f \in H^\infty(\mathbb{B}_n, X).$$

In the sequel, we will assume that the symbol b satisfies the following condition

$$\int_{\mathbb{B}_n} \frac{\|b(w)\|_{\mathcal{L}(\overline{X},Y)}}{|1-\langle z, w\rangle|^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) < \infty, \qquad \text{for every } \ z \in \mathbb{B}_n. \tag{6.1.4.1}$$

It is easy to check that if b satisfies (6.1.4.1), then the little Hankel operator h_b is well defined on $H^\infty(\mathbb{B}_n, X)$.

6.1.5 Problems and known results

The boundedness properties of the little Hankel operator in the classical case (that is, when $X = Y = \mathbb{C}$) have been extensively studied and many results are now well known. For the case $n = 1$, important references are [89] and [99]. For $n > 1$, a complete characterization has been obtained by Aline Bonami and Luo Luo in [87] when $p \leq q$. In 2015, Pau and Zhao [95] solved the case $1 < q < p < \infty$. Indeed, they showed that if b is a holomorphic symbol, the little Hankel operator h_b extends to a bounded operator from $A^p_\alpha(\mathbb{B}_n, \mathbb{C})$ into $A^q_\alpha(\mathbb{B}_n, \mathbb{C})$, with $1 < q < p < \infty$, if and only if the symbol b belongs to the weighted Bergman space $A^t_\alpha(\mathbb{B}_n, \mathbb{C})$ where $1/t = 1/q - 1/p$. We are here concerned with the question of characterizing the operator-valued holomorphic symbols b for which the little Hankel operator h_b extends into a bounded operator from $A^p_\alpha(\mathbb{B}_n, X)$ into $A^q_\alpha(\mathbb{B}_n, Y)$ where $0 < p, q < \infty$. In [84] Aleman and Constantin solved this problem for the particular case $n = 1$, $p = q = 2$ and $X = Y = \mathcal{H}$ where $\mathcal{H}$ is a separable Hilbert space. They showed that the little Hankel operator h_b extends into a bounded operator from $A^2_\alpha(\mathbb{B}_n, \mathcal{H})$ into $A^2_\alpha(\mathbb{B}_n, \mathcal{H})$ if and only if the symbol b belongs to the Bloch space $\mathcal{B}(\mathbb{B}_n, \mathcal{L}(\mathcal{H}))$. Constantin also obtained in [88] that the little Hankel operator h_b is a compact operator from $A^2_\alpha(\mathbb{B}_n, \mathcal{H})$ into $A^2_\alpha(\mathbb{B}_n, \mathcal{H})$ if and only if the symbol b belongs to the little vector-valued Bloch space $\mathcal{B}_0(\mathbb{B}_n, \mathcal{K}(\mathcal{H}))$. Their results extend clearly the one known in the classical case (when $\mathcal{H} = \mathbb{C}$). In [94], Oliver solved this problem in the case $1 < p, q < \infty$. Mainly, he showed that for $1 < p < \infty$, the little Hankel operator h_b is bounded from $A^p_\alpha(\mathbb{B}_n, X)$ into $A^p_\alpha(\mathbb{B}_n, Y)$ if and only if the symbol b belongs to the vector-valued Bloch space $\mathcal{B}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ and this result clearly generalizes the one obtained by Aleman and Constantin in [84]. Moreover, for $1 < p < q < \infty$, Oliver showed that the little Hankel operator h_b is bounded from $A^p_\alpha(\mathbb{B}_n, X)$ into $A^q_\alpha(\mathbb{B}_n, Y)$ if and only if the symbol b belongs to the γ-Bloch space $\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ with $\gamma = 1 + (n+1+\alpha)\left(\frac{1}{q} - \frac{1}{p}\right)$. Also for $1 < q < p < \infty$, Oliver showed that the little Hankel operator h_b is bounded from $A^p_\alpha(\mathbb{B}_n, X)$ into $A^q_\alpha(\mathbb{B}_n, Y)$ if and only if $b \in A^t_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, with $1/t = 1/q - 1/p$, which generalizes the main result in [95]. We are also concerned here with the question of characterizing the operator-valued holomorphic symbols for which h_b extends into a compact operator from $A^p_\alpha(\mathbb{B}_n, X)$ into $A^q_\alpha(\mathbb{B}_n, Y)$ where $1 < p \leq q < \infty$.

6.1.6 Topological duals

Let X be a complex Banach space and $0 < p \leq 1$. The topological dual of the Bergman space $A^p_\alpha(\mathbb{B}_n, X)$ can be identified with the Lipschitz space $\Gamma_\gamma(\mathbb{B}_n, X^\star)$ as follows:

Theorem 6.1.3. *Let $0 < p \leq 1$. The space $(A^p_\alpha(\mathbb{B}_n, X))^\star$ can be identified with $\Gamma_\gamma(\mathbb{B}_n, X^\star)$ with $\gamma = (n + 1 + \alpha)\left(\frac{1}{p} - 1\right)$ under the pairing*

$$\langle f, g \rangle_{\alpha,X} = c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z) \rangle_{X,X^\star} (1 - |z|^2)^k \mathrm{d}\nu_\alpha(z), \tag{6.1.6.1}$$

where D_k is defined by (6.2.1.1), *$k > \gamma$, is an integer, $g \in \Gamma_\gamma(\mathbb{B}_n, X^\star)$ and $f \in A^p_\alpha(\mathbb{B}_n, X)$. Moreover,*

$$\|g\|_{\Gamma_\gamma(\mathbb{B}_n, X^\star)} \simeq \sup_{\|f\|_{A^p_\alpha(\mathbb{B}_n, X)} = 1} |\langle f, g \rangle_{\alpha,X}|.$$

Before stating the next results, we need to make another assumption on the operator-valued symbol b. More precisely, we assume that the operator-valued holomorphic symbol b satisfies the following condition:

$$\int_{\mathbb{B}_n} \|b(z)\|_{\mathcal{L}(\overline{X}, Y)} \log\left(\frac{1}{1 - |z|^2}\right) \mathrm{d}\nu_\alpha(z) < \infty. \tag{6.1.6.2}$$

Let X and Y be two complex Banach spaces. Our contributions to the boundedness problem of the little Hankel operator with operator-valued symbol for $0 < p, q \leq 1$ are the following :

Theorem 6.1.4. *Suppose $0 < p \leq 1$, and $\alpha > -1$. If the little Hankel operator h_b extends to a bounded operator from $A^p_\alpha(\mathbb{B}_n, X)$ into $A^q_\alpha(\mathbb{B}_n, Y)$ for some positive $q < 1$, then the symbol b is in $\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ with $\gamma = (n+1+\alpha)\left(\frac{1}{p} - 1\right)$. Conversely, if b is in $\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ with $\gamma = (n+1+\alpha)\left(\frac{1}{p} - 1\right)$, then the little Hankel operator $h_b : A^p_\alpha(\mathbb{B}_n, X) \longrightarrow A^{1,\infty}_\alpha(\mathbb{B}_n, Y)$ is a bounded operator.*

As a direct consequence, we have the following result:

Corollary 6.1.5. *Suppose $0 < p \leq 1$, and $\alpha > -1$. The little Hankel operator h_b extends to a bounded operator from $A^p_\alpha(\mathbb{B}_n, X)$ into $A^q_\alpha(\mathbb{B}_n, Y)$ for some positive $q < 1$ if and only if its symbol b belongs to $\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, where $\gamma = (n+1+\alpha)\left(\frac{1}{p} - 1\right)$.*

Theorem 6.1.6. *Let $0 < p \leq 1$, $\alpha > -1$ and $\gamma = (n+1+\alpha)\left(\frac{1}{p} - 1\right)$. The little Hankel operator extends to a bounded operator from $A^p_\alpha(\mathbb{B}_n, X)$ into $A^1_\alpha(\mathbb{B}_n, Y)$ if and only if for some integer $k > \gamma$,*

$$\|N^k b(w)\|_{\mathcal{L}(\overline{X}, Y)} \leq \frac{C}{(1-|w|^2)^{k-\gamma}} \left(\log \frac{1}{1-|w|^2}\right)^{-1} \qquad w \in \mathbb{B}_n. \quad (6.1.6.3)$$

Theorem 6.1.7. *Suppose $1 < p \leq q < \infty$. The little Hankel operator h_b : $A^p_\alpha(\mathbb{B}_n, X)$*

$\to A^q_\alpha(\mathbb{B}_n, Y)$ is a bounded operator if and only if $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, where

$$\gamma_0 = (n+1+\alpha)\left(\frac{1}{p} - \frac{1}{q}\right).$$

Moreover, $\|h_b\|_{A^p_\alpha(\mathbb{B}_n, X) \to A^q_\alpha(\mathbb{B}_n, Y)} \simeq \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))}$.

If X, Y are reflexive complex Banach spaces, then we have the following theorem

Theorem 6.1.8. *Suppose that $1 < p \leq q < \infty$, and $\alpha > -1$ The little Hankel operator $h_b : A^p_\alpha(\mathbb{B}_n, X) \longrightarrow A^q_\alpha(\mathbb{B}_n, Y)$ is a compact operator if and only if*

$$b \in \Lambda_{\gamma_0,0}(\mathbb{B}_n, \mathcal{K}(\overline{X}, Y)),$$

where $\Lambda_{\gamma_0,0}(\mathbb{B}_n, \mathcal{K}(\overline{X}, Y))$ denotes the generalized little vector-valued Lipschitz space and $\gamma_0 = (n+1+\alpha)\left(\frac{1}{p} - \frac{1}{q}\right)$, see (6.1.3.3).

6.1.7 Vector-valued holomorphic functions

In Section 2, we recall some preliminary notions on vector-valued holomorphic functions and we also give the proofs of some important results. Section 3 contains the proof of Theorem 6.1.3 on the dual of the vector-valued Bergman space $A^p_\alpha(\mathbb{B}_n, X)$ for $0 < p \leq 1$. In Section 4, we give the proof of Theorem 6.1.4 and Corollary 6.1.5. In Section 5, we give the proof of Theorem 6.1.6. In Section 6, We first give some preliminaries results to prepare the proof of Theorem 6.1.8. We recall the result by Oliver [94] of the boundedness of the little Hankel operator with operator-valued symbol h_b from $A^p_\alpha(\mathbb{B}_n, X)$ into $A^q_\alpha(\mathbb{B}_n, Y)$, with $1 < p \leq q < \infty$ and we generalize it. In the same section, we give the proof of Theorem 6.1.8.

Throughout this chater, when there is no additional condition, X and Y will denotes two complex Banach spaces, the real parameter α will be chosen such that $\alpha > -1$ and c will be a positive constant whose value may change from one occurrence to the next. We will also adopt the following notation: we will write $A \lesssim B$ whenever there exists a positive constant c such that $A \leq cB$. We also write $A \simeq B$ when $A \lesssim B$ and $B \lesssim A$.

6.2 Vector-valued Bergman projection and integral estimates

Here we give some definitions and notations which will be used later and can be found in [94] and [87].

For $f \in L^1_\alpha(\mathbb{B}_n, X)$ and $z \in \mathbb{B}_n$, the Bergman projection $P_\alpha f$ of f is the integral operator defined by

$$P_\alpha f(z) := \int_{\mathbb{B}_n} K_\alpha(z,w) f(w) \mathrm{d}\nu_\alpha(w),$$

where $K_\alpha(z,w) := \dfrac{1}{(1-\langle z, w\rangle)^{n+1+\alpha}}$ is the Bergman reproducing kernel of $\mathbb{B}_n$. In this situation, $P_\alpha f$ is also a X-valued holomorphic function.

Lemma 6.2.1 (Density). *Suppose that $0 < p < \infty$. Then the space of all bounded vector-valued holomorphic functions $H^\infty(\mathbb{B}_n, X)$ is dense in $A^p_\alpha(\mathbb{B}_n, X)$.*

Proof. We are going to give the proof for $0 < p < 1$, since the case $1 \leq p < \infty$ is [94, Lemma 2.1.4]. Given a function $f \in A^p_\alpha(\mathbb{B}_n, X)$, let f_ρ defined for $z \in \mathbb{B}_n$ by $f_\rho(z) := f(\rho z)$, where $0 < \rho < 1$. The function f_ρ is holomorphic in the set $\{z \in \mathbb{B}_n : |z| < 1/\rho\}$ hence is bounded on $\mathbb{B}_n$. We first recall that the integral means

$$M_p(r,f) := \int_{\mathbb{S}_n} \|f(r\zeta)\|^p_X \mathrm{d}\sigma(\zeta), \qquad 0 \leq r < 1$$

are increasing with r, see [98, Corollary 4.21]. Since $M_p(r, f_\rho) = M_p(\rho r, f)$, we have by Minkowski's inequality that

$$M^p_p(r, f_\rho - f) \leq M^p_p(r,f) + M^p_p(r, f_\rho) \leq 2M^p_p(r,f).$$

By the formula of [94, (1.1.1)], (integration in polar coordinates formula) we get

$$\|f - f_\rho\|^p_{p,\alpha,X} = 2nc_\alpha \int_0^1 M^p_p(r, f_\rho - f)(1-r^2)^\alpha r^{2n-1} \mathrm{d}r. \tag{6.2.0.1}$$

Since $f \in A^p_\alpha(\mathbb{B}_n, X)$, we have that the function $M^p_p(r, f)$ is integrable over the interval $[0, 1)$ with respect to the measure $2n(1-r^2)^\alpha r^{2n-1}\mathrm{d}r$. It is also clear that $f_\rho \to f$ on any compact subsets of $\mathbb{B}_n$ which implies that $M^p_p(r, f_\rho - f) \to 0$ for each $r \in [0, 1)$ as $\rho \to 1$. Applying the dominated convergence theorem in (6.2.0.1), we obtain that $\|f - f_\rho\|^p_{p,\alpha,X} \longrightarrow 0$, as $\rho \to 1$. □

Corollary 6.2.2. *For $0 < p \leq 1$, the following inclusion is dense*

$$A^2_\alpha(\mathbb{B}_n, X) \subset A^p_\alpha(\mathbb{B}_n, X).$$

Proof. The proof follows directly from Lemma 6.2.1. □

In [86], Oscar Blasco obtained the duality theorem for the vector-valued Bergman spaces in the unit disc $\mathbb{B}_1$ without any restriction on the Banach space. The proof also works for the unit ball $\mathbb{B}_n$. The result is stated as follows:

Theorem 6.2.3 (Duality). *Suppose $1 < p < \infty$. The dual space $(A^p_\alpha(\mathbb{B}_n, X))^\star$ can be identified with $A^{p'}_\alpha(\mathbb{B}_n, X^\star)$, where p' is the conjugate exponent of p given by $\frac{1}{p} + \frac{1}{p'} = 1$, under the integral pairing defined by*

$$\langle f, g\rangle_{\alpha,X} := \int_{\mathbb{B}_n} \langle f(z), g(z)\rangle_{X,X^\star} \mathrm{d}\nu_\alpha(z), \tag{6.2.0.2}$$

for any $f \in A^p_\alpha(\mathbb{B}_n, X)$, $g \in A^{p'}_\alpha(\mathbb{B}_n, X^\star)$.

Remark 6.2.4. Suppose $1 < p < \infty$. If X is a reflexive complex Banach space, then the vector-valued Bergman space $A^p_\alpha(\mathbb{B}_n, X)$ is a reflexive Banach space.

The following reproducing kernel formula also holds for vector-valued Bergman spaces. The proof can be found in [94, Proposition 2.1.2].

Proposition 6.2.5. *Let $f \in A^1_\alpha(\mathbb{B}_n, X)$. We have*

$$f(z) := \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w),$$

for any $z \in \mathbb{B}_n$.

We have the following pointwise estimate on the vector-valued Bergman spaces. The proof can be found in [94].

Theorem 6.2.6. *Let* $0 < p < \infty$. *Then*

$$\|f(z)\|_X \leq \frac{\|f\|_{p,\alpha,X}}{(1-|z|^2)^{(n+1+\alpha)/p}},$$

for any $f \in A^p_\alpha(\mathbb{B}_n, X)$ *and* $z \in \mathbb{B}_n$.

The following lemma is critical for many problems concerning the weighted vector-valued Bergman spaces $A^p_\alpha(\mathbb{B}_n, X)$ whenever $0 < p \leq 1$ and will be extensively used.

Lemma 6.2.7. *Let* $0 < p \leq 1$. *Then*

$$\int_{\mathbb{B}_n} \|f(z)\|_X (1-|z|^2)^{(\frac{1}{p}-1)(n+1+\alpha)} \mathrm{d}\nu_\alpha(z) \leq \|f\|_{p,\alpha,X},$$

for all $f \in A^p_\alpha(\mathbb{B}_n, X)$.

Proof. Write

$$\|f(z)\|_X = \|f(z)\|_X^p \|f(z)\|_X^{1-p},$$

and estimate the second factor using Theorem 6.2.6. The desired result follows. □

The following technical result is proved in [87, Lemma 3.1]

Lemma 6.2.8. *Let* $\beta, \delta > 0$. *For all* $w \in \mathbb{B}_n$, *we have*

$$I_\alpha(w) := \int_{\mathbb{B}_n} \left|\log\left(\frac{1-\langle z,w\rangle}{1-|w|^2}\right)\right|^\delta \frac{(1-|w|^2)^\beta}{|1-\langle z,w\rangle|^{n+1+\alpha+\beta}} \mathrm{d}\nu_\alpha(z) \leq C,$$

where C *is independent of* w *and* $\log$ *is the principal branch of the logarithm.*

In the sequel, we will also need the following lemma which the scalar version can be found in [91].

Lemma 6.2.9. *If $0 < q < 1$, then the identity $i : L^{1,\infty}_\alpha(\mathbb{B}_n, X) \hookrightarrow L^q_\alpha(\mathbb{B}_n, X)$ is continuous in the sense that there exists a constant $C(q) > 0$ such that for every $f \in L^{1,\infty}_\alpha(\mathbb{B}_n, X)$, we have*

$$\|f\|_{q,\alpha,X} \leq C(q)\|f\|_{L^{1,\infty}_\alpha(\mathbb{B}_n,X)}.$$

The following result will be very useful in many situations. A proof can be found in [98].

Theorem 6.2.10. *For $\beta \in \mathbb{R}$, let*

$$I_{\alpha,\beta}(z) := \int_{\mathbb{B}_n} \frac{(1-|w|^2)^\alpha \mathrm{d}\nu(w)}{|1-\langle z, w\rangle|^{n+1+\alpha+\beta}}, \qquad z \in \mathbb{B}_n.$$

(i) If $\beta = 0$, there exists a constant $C > 0$ such that

$$I_{\alpha,\beta}(z) \leq C \log \frac{1}{1-|z|^2}, \quad z \in \mathbb{B}_n.$$

(ii) If $\beta > 0$, there exists a constant $C > 0$ such that

$$I_{\alpha,\beta}(z) \leq C \frac{1}{(1-|z|^2)^\beta}, \quad z \in \mathbb{B}_n.$$

(iii) If $\beta < 0$, there exists a constant $C > 0$ such that

$$I_{\alpha,\beta}(z) \leq C.$$

6.2.1 Differential operators and equivalent norms for Γ_γ

Given a positive integer k, we define the differential operator D_k by

$$D_k := (2I + N) \circ (3I + N) \circ \ldots \circ ((k+1)I + N), \tag{6.2.1.1}$$

where I is the identity operator and N is the differential operator given in (6.1.3.1).

In the sequel, we denote by $\mathcal{P}(\mathbb{B}_n, X)$ the space of all vector-valued holomorphic polynomials. The proof of the following lemma is similar as in the scalar case in [93].

Lemma 6.2.11. *For all $f \in \mathcal{P}(\mathbb{B}_n, X)$ and $g \in \mathcal{P}(\mathbb{B}_n, X^\star)$, we have the following identity*

$$\int_{\mathbb{B}_n} \langle f(z), g(z)\rangle_{X,X^\star} \mathrm{d}\nu_\alpha(z) \quad = \quad c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z)\rangle_{X,X^\star} (1-|z|^2)^k \mathrm{d}\nu_\alpha(z),$$

where c_k is a positive constant depending only on the integer k. The above identities are valid for vector-valued holomorphic functions when both sides make sense.

The following lemma will be very useful in the sequel.

Lemma 6.2.12. *Let $\{a_k\}$ a sequence of positive numbers. For any positive integer k, let M_k the differential operator of order k defined by*

$$M_k := (a_0 I + N) \circ (a_1 I + N) \circ \ldots \circ (a_{k-1} I + N).$$

Then a vector-valued holomorphic function f belongs to $\Gamma_\gamma(\mathbb{B}_n, X)$ if and only if there exists an integer $k > \gamma$ such that

$$\sup_{z \in \mathbb{B}_n} (1-|z|^2)^{k-\gamma} \|M_k f(z)\|_X < \infty.$$

Proof. Let us assume first that $f \in \Gamma_\gamma(\mathbb{B}_n, X)$, and we prove the desired estimate on M_k. By assumption, there exists an integer $k > \gamma$ and a positive constant C such that

$$\|N^k f(z)\|_X \leq C(1-|z|^2)^{\gamma-k},$$

for any $z \in \mathbb{B}_n$. It is enough to prove that the following inequality

$$\|N^j f(z)\|_X < C(1-|z|^2)^{\gamma-k},$$

holds for $0 \leq j < k$, since the assumption give the case $j = k$. For $g \in \mathcal{H}(\mathbb{B}_n, X)$ and $z = rz'$, where $r = |z|$, and z' is in the unit sphere. We have

$$Ng(rz') = r\partial_r g(rz').$$

Thus,

$$g(rz') - g(z'/2) = \int_{\frac{1}{2}}^{r} Ng(sz')\frac{ds}{s}.$$

Now, for $g \in \mathcal{H}(\mathbb{B}_n, X)$ such that $\|Ng(z)\|_X \leq C(1-|z|^2)^{\gamma-k}$. We have that

$$\begin{aligned}
\|g(rz') - g(z'/2)\|_X &\leq 2\int_{\frac{1}{2}}^{r} \|Ng(sz')\|_X \mathrm{d}s \\
&\leq 4C\int_{\frac{1}{2}}^{r} (1-s^2)^{\gamma-k} s\mathrm{d}s \\
&= -2C\int_{\frac{1}{2}}^{r} -2s(1-s^2)^{\gamma-k}\mathrm{d}s \\
&= \left[\frac{-2C}{\gamma-k+1}(1-s^2)^{\gamma-k+1}\right]_{\frac{1}{2}}^{r} \\
&= \frac{-2C}{\gamma-k+1}\left\{(1-r^2)^{\gamma-k+1} - (1-\frac{1}{4})^{\gamma-k+1}\right\}.
\end{aligned}$$

Now, if $\gamma - k + 1 < 0$, then

$$\|g(rz') - g(z'/2)\|_X \leq \frac{-2C}{\gamma-k+1}(1-r^2)^{\gamma-k} = C_{k,\gamma}(1-|z|^2)^{\gamma-k}.$$

If $\gamma - k + 1 > 0$, then

$$\begin{aligned}
\|g(rz') - g(z'/2)\|_X &\leq \frac{2C}{\gamma-k+1}\left\{(1-\frac{1}{4})^{\gamma-k+1} - (1-r^2)^{\gamma-k+1}\right\} \\
&\leq \frac{2C}{\gamma-k+1}(1-\frac{1}{4})^{\gamma-k+1} = C'_{k,\gamma} \\
&\leq C'_{k,\gamma}(1-|z|^2)^{\gamma-k},
\end{aligned}$$

where the last inequality is justified using the fact that $(1-|z|^2)^{\gamma-k} > 1$. It then follows that

$$\|g(z)\|_X \leq C(1-|z|^2)^{\gamma-k}.$$

Now, we use this fact inductively for $g = N^k f$, then $g = N^{k-1} f, \quad \cdots$ to conclude. Conversely, assume that there exists an integer $k > \gamma$ and a positive constant C such that

$$\|M_k f(z)\|_X \leq C(1 - |z|^2)^{\gamma - k},$$

for any $z \in \mathbb{B}_n$. To conclude, it is sufficient to prove that for a fixed positive real a, the inequality

$$\|ag(z) + Ng(z)\|_X \leq C(1 - |z|^2)^{\gamma - k} \tag{6.2.1.2}$$

implies the inequality

$$\|Ng(z)\|_X \leq C(1 - |z|^2)^{\gamma - k},$$

for any function $g \in \mathcal{H}(\mathbb{B}_n, X)$. Choose a real β such that $\beta + \gamma - k > -1$. By the assumption (6.2.1.2), we have that

$$\int_{\mathbb{B}_n} \|ag(z) + Ng(z)\|_X (1 - |z|^2)^\beta \mathrm{d}\nu(z) < \infty.$$

Thus, for any $z \in \mathbb{B}_n$, we have

$$ag(z) + Ng(z) = c_\beta \int_{\mathbb{B}_n} \frac{[ag(w) + Ng(w)]}{(1 - \langle z, w \rangle)^{n+1+\beta}} (1 - |w|^2)^\beta \mathrm{d}\nu(w).$$

Then, differentiating under the integral sign, we obtain that for all $1 \leq i \leq n$, we get

$$\partial_{z_i} [ag(z) + Ng(z)] = (n + 1 + \beta) c_\beta \int_{\mathbb{B}_n} \frac{[ag(w) + Ng(w)]\overline{w}_i}{(1 - \langle z, w \rangle)^{n+2+\beta}} (1 - |w|^2)^\beta \mathrm{d}\nu(w).$$

Therefore,

$$N (ag(z) + Ng(z)) = (n + 1 + \beta) c_\beta \int_{\mathbb{B}_n} \frac{[ag(w) + Ng(w)]\langle z, w \rangle}{(1 - \langle z, w \rangle)^{n+2+\beta}} (1 - |w|^2)^\beta \mathrm{d}\nu(w).$$

Applying (6.2.1.2), and Theorem 6.2.10, we get that for all $1 \leq i \leq n$,

$$\begin{aligned} \| N\left(ag(z) + Ng(z)\right)\|_X &\leq Cc_\beta \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{\gamma-k+\beta}}{|1-\langle z,w\rangle|^{n+1+\gamma-k+\beta+(k-\gamma+1)}} \mathrm{d}\nu(w) \\ &\leq C(1-|z|^2)^{\gamma-k-1}. \end{aligned}$$

Thus, the derivative of $ag(z) + Ng(z)$ is bounded by $(1-|z|^2)^{\gamma-k-1}$. So, to prove the inequality above, we are reduced to consider smooth functions ϕ of one variable $r \in [0,1)$, and to prove that the inequality

$$\|\psi'(r)\|_X \leq C(1-r)^{\gamma-k-1},$$

with $\psi(r) = a\phi(r) + r\phi'(r)$, implies that

$$\|r\phi'(r)\|_X \leq C(1-r)^{\gamma-k}$$

(here, $\phi(r) = g(rz')$). Now, differentiating ψ, we obtain $\psi'(s) = (a+1)\phi'(s) + s\phi''(s)$. Multiplying both sides of the previous inequality by s^a, we obtain that $s^a\psi'(s) = (a+1)s^a\phi'(s) + s^{a+1}\phi''(s) = \left[s^{a+1}\phi'(s)\right]'$. Then integrating the equality above on $[0,r]$, we obtain that

$$\phi'(r) = \frac{1}{r^{a+1}} \int_0^r s^a\psi'(s)\mathrm{d}s.$$

Therefore, the desired estimate follows at once, since $k > \gamma$. □

Remark 6.2.13. We shall use extensively this lemma for two particular classes of differential operators: first the class D_k, then the class L_k, corresponding to the choice $a_j = n + \alpha + j + 1$. For this choice, we have

$$(a_jI + N)(1-\langle z,w\rangle)^{-n-\alpha-j-1} = \frac{n+\alpha+j+1}{(1-\langle z,w\rangle)^{n+\alpha+j+2}},$$

and inductively,

$$L_k(1-\langle z,w\rangle)^{-n-\alpha-1} = \frac{c_k}{(1-\langle z,w\rangle)^{n+\alpha+k+1}}.$$

The proof of Lemma 6.2.12 allows us to define an equivalent norm of f in terms of $M_k f$. Particularly, we will write the equivalent norms of f in terms of $D_k f$ and $L_k f$. More precisely, we have the following result:

Corollary 6.2.14. *Let D_k a differential operator of order k defined in* (6.2.1.1) *and L_k a differential operator of order k defined in Remark* 6.2.13. *For vector-valued holomorphic functions, the following assertions are equivalent:*

(1) $f \in \Gamma_\gamma(\mathbb{B}_n, X)$.

(2) *There exists an integer $k > \gamma$ such that*

$$\sup_{z\in\mathbb{B}_n} (1-|z|^2)^{k-\gamma} \|D_k f(z)\|_X < \infty.$$

(3) *There exists an integer $k > \gamma$ such that*

$$\sup_{z\in\mathbb{B}_n} (1-|z|^2)^{k-\gamma} \|L_k f(z)\|_X < \infty.$$

Moreover, the following are equivalent

$$\begin{aligned} \|f\|_{\Gamma_\gamma(\mathbb{B}_n,X)} &\simeq \|f(0)\|_X + \sup_{z\in\mathbb{B}_n} (1-|z|^2)^{k-\gamma} \|D_k f(z)\|_X \\ &\simeq \|f(0)\|_X + \sup_{z\in\mathbb{B}_n} (1-|z|^2)^{k-\gamma} \|L_k f(z)\|_X. \end{aligned}$$

The proof of some of the results obtained in this paper will be based on the following lemma. A proof is in [94], but for the sake of completeness, we will recall the proof.

Lemma 6.2.15. *Let $f \in H^\infty(\mathbb{B}_n, X)$ and $g \in H^\infty(\mathbb{B}_n, Y^\star)$. If $b \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ is such that* (6.1.4.1) *and* (6.1.6.2) *hold. Then we have*

$$\langle h_b f, g\rangle_{\alpha,Y} = \int_{\mathbb{B}_n} \langle b(z)\overline{f(z)}, g(z)\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z). \tag{6.2.1.3}$$

Proof. Let $f \in H^\infty(\mathbb{B}_n, X)$ and $g \in H^\infty(\mathbb{B}_n, Y^\star)$. By the definition of $\langle\cdot,\cdot\rangle_{\alpha,Y}$, Fubini's theorem, Lemma 6.1.1 and the reproducing kernel property, we have:

$$\begin{aligned}
\langle h_b(f), g\rangle_{\alpha,Y} &= \int_{\mathbb{B}_n} \langle h_b(f)(z), g(z)\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z)\\
&= \int_{\mathbb{B}_n} \langle \int_{\mathbb{B}_n} \frac{b(w)(\overline{f(w)})\mathrm{d}\nu_\alpha(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}, g(z)\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z)\\
&= \int_{\mathbb{B}_n} g(z)\left(\int_{\mathbb{B}_n} \frac{b(w)(\overline{f(w)})\mathrm{d}\nu_\alpha(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}\right)\mathrm{d}\nu_\alpha(z)\\
&= \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} g(z)\left(\frac{b(w)(\overline{f(w)})}{(1-\langle z,w\rangle)^{n+1+\alpha}}\right)\mathrm{d}\nu_\alpha(w)\mathrm{d}\nu_\alpha(z)\\
&= \int_{\mathbb{B}_n}\left(\int_{\mathbb{B}_n} \frac{g(z)}{(1-\langle w,z\rangle)^{n+1+\alpha}}\mathrm{d}\nu_\alpha(z)\right)\left(b(w)(\overline{f(w)})\right)\mathrm{d}\nu_\alpha(w)\\
&= \int_{\mathbb{B}_n} g(w)\left(b(w)(\overline{f(w)})\right)\mathrm{d}\nu_\alpha(w)\\
&= \int_{\mathbb{B}_n} \langle b(w)\overline{f(w)}, g(w)\rangle_{Y,Y^\star}\mathrm{d}\nu_\alpha(w).
\end{aligned}$$

It remains to show that the assumption of Fubini's theorem is fulfilled. Indeed, since $f \in H^\infty(\mathbb{B}_n, X)$ and $g \in H^\infty(\mathbb{B}_n, Y^\star)$, by Tonelli's theorem, Theorem 6.2.10 and relation (6.1.6.2) we have that

$$\begin{aligned}
\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\left|\frac{g(z)\left(b(w)(\overline{f(w)})\right)}{(1-\langle z,w\rangle)^{n+1+\alpha}}\right|\mathrm{d}\nu_\alpha(w)\mathrm{d}\nu_\alpha(z) &\lesssim \int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{\|b(w)\|_{\mathcal{L}(\overline{X},Y)}}{|1-\langle z,w\rangle|^{n+1+\alpha}}\mathrm{d}\nu_\alpha(w)\\
&\lesssim \int_{\mathbb{B}_n}\|b(w)\|_{\mathcal{L}(\overline{X},Y)}\log\left(\frac{1}{1-|w|^2}\right)
\end{aligned}$$

□

Lemma 6.2.16. *Let $f \in H^\infty(\mathbb{B}_n, X)$ and $z \in \mathbb{B}_n$. For $b \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ satisfying* (6.1.4.1) *and* (6.1.6.2), *the function*

$$g_z(w) := \frac{f(w)}{(1-\langle w,z\rangle)^{n+1+\alpha}}, \qquad w \in \mathbb{B}_n$$

belongs to $H^\infty(\mathbb{B}_n, X)$ and the following identity holds:

$$h_b(f)(z) = C_k \int_{\mathbb{B}_n} L_k\left(b(w)(\overline{g_z(w)})\right)\mathrm{d}\nu_{\alpha+k}(w),$$

where k is any positive integer and C_k is a positive constant depending only on k.

Proof. It is clear that $g_z \in H^\infty(\mathbb{B}_n, X)$. By the definition of the little Hankel operator and the reproducing kernel property, we have

$$\begin{aligned}
h_b(f)(z) &= \int_{\mathbb{B}_n} \frac{b(w)\overline{f(w)}}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} b(w) \left(\overline{\frac{f(w)}{(1-\langle w,z\rangle)^{n+1+\alpha}}}\right) \mathrm{d}\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} b(w)(\overline{g_z(w)}) \mathrm{d}\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} b(w) \left(\overline{\int_{\mathbb{B}_n} \frac{g_z(\zeta)}{(1-\langle w,\zeta\rangle)^{n+1+\alpha+k}} \mathrm{d}\nu_{\alpha+k}(\zeta)}\right) \mathrm{d}\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{b(w)(\overline{g_z(\zeta)})}{(1-\langle \zeta,w\rangle)^{n+1+\alpha+k}} \mathrm{d}\nu_\alpha(w)\right) \mathrm{d}\nu_{\alpha+k}(\zeta) \\
&= c_k^{-1} \int_{\mathbb{B}_n} L_k \left(\int_{\mathbb{B}_n} \frac{b(w)(\overline{g_z(\zeta)})}{(1-\langle \zeta,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w)\right) \mathrm{d}\nu_{\alpha+k}(\zeta) \\
&= c_k^{-1} \int_{\mathbb{B}_n} L_k \left(b(\zeta)(\overline{g_z(\zeta)})\right) \mathrm{d}\nu_{\alpha+k}(\zeta).
\end{aligned}$$

The assumption of Fubini's theorem is fulfilled. Indeed by (6.1.6.2), we have that

$$\int_{\mathbb{B}_n} \left\| \int_{\mathbb{B}_n} \frac{b(w)(\overline{g_z(\zeta)})}{(1-\langle \zeta,w\rangle)^{n+1+\alpha+k}} \mathrm{d}\nu_\alpha(w) \right\|_Y \mathrm{d}\nu_{\alpha+k}(\zeta)$$

$$\begin{aligned}
&\leq \|g_z\|_{\infty,X} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{\|b(w)\|_{\mathcal{L}(\overline{X},Y)}}{|1-\langle w,\zeta\rangle|^{n+1+\alpha+k}} \mathrm{d}\nu_\alpha(w) \mathrm{d}\nu_{\alpha+k}(\zeta) \\
&= \|g_z\|_{\infty,X} \int_{\mathbb{B}_n} \|b(w)\|_{\mathcal{L}(\overline{X},Y)} \\
&\times \left(\int_{\mathbb{B}_n} \frac{\mathrm{d}\nu_{\alpha+k}(\zeta)}{|1-\langle w,\zeta\rangle|^{n+1+\alpha+k}}\right) \mathrm{d}\nu_\alpha(w) \\
&\leq \|g_z\|_{\infty,X} \int_{\mathbb{B}_n} \|b(w)\|_{\mathcal{L}(\overline{X},Y)} \left(\log \frac{1}{1-|w|^2}\right) \mathrm{d}\nu_\alpha(w) \\
&< \infty.
\end{aligned}$$

$\square$

6.3 The Proof of Theorem 6.1.3

Proof. We first suppose that $g \in \Gamma_\gamma(\mathbb{B}_n, X^\star)$, with $\gamma = (n+1+\alpha)\left(\frac{1}{p}-1\right)$. Given a positive integer $k > \gamma$, we define the functional

$$\wedge_g : A^p_\alpha(\mathbb{B}_n, X) \longrightarrow \mathbb{C}$$

$$f \mapsto \wedge_g(f) = c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z)\rangle_{X,X^\star} (1-|z|^2)^k \mathrm{d}\nu_\alpha(z),$$

where c_k is the positive constant in Lemma 6.2.11. It is clear that $\wedge_g$ is linear and is well defined on $A^p_\alpha(\mathbb{B}_n, X)$. Indeed, let $f \in A^p_\alpha(\mathbb{B}_n, X)$. By Lemma 6.2.7, we have

$$\begin{aligned}
|\wedge_g(f)| &= c_k \left| \int_{\mathbb{B}_n} \langle f(z), D_k g(z)\rangle_{X,X^\star} (1-|z|^2)^k \mathrm{d}\nu_\alpha(z) \right| \\
&\leq c_k \int_{\mathbb{B}_n} \|f(z)\|_X \|D_k g(z)\|_{X^\star} (1-|z|^2)^k \mathrm{d}\nu_\alpha(z) \\
&= c_k \int_{\mathbb{B}_n} (1-|z|^2)^{k-\gamma} \|D_k g(z)\|_{X^\star} (1-|z|^2)^\gamma \|f(z)\|_X \mathrm{d}\nu_\alpha(z) \\
&\leq c_k \sup_{z \in \mathbb{B}_n} (1-|z|^2)^{k-\gamma} \|D_k g(z)\|_{X^\star} \int_{\mathbb{B}_n} (1-|z|^2)^\gamma \|f(z)\|_X \mathrm{d}\nu_\alpha(z) \\
&\lesssim \|g\|_{\Gamma_\gamma(\mathbb{B}_n, X^\star)} \int_{\mathbb{B}_n} (1-|z|^2)^{(\frac{1}{p}-1)(n+1+\alpha)} \|f(z)\|_X \mathrm{d}\nu_\alpha(z) \\
&\lesssim \|g\|_{\Gamma_\gamma(\mathbb{B}_n, X^\star)} \|f\|_{p,\alpha,X}.
\end{aligned}$$

We conclude that $\wedge_g$ is bounded on $A^p_\alpha(\mathbb{B}_n, X)$ and $\|\wedge_g\| \lesssim \|g\|_{\Gamma_\gamma(\mathbb{B}_n, X^\star)}$.

Conversely, let $\wedge$ be a bounded linear functional on $A^p_\alpha(\mathbb{B}_n, X)$. Let us show that there exists $g \in \Gamma_\gamma(\mathbb{B}_n, X^\star)$, with $\gamma = (n+1+\alpha)\left(\frac{1}{p}-1\right)$ such that $\wedge = \wedge_g$. Since $A^2_\alpha(\mathbb{B}_n, X) \subset A^p_\alpha(\mathbb{B}_n, X)$ and $\wedge$ is bounded on $A^p_\alpha(\mathbb{B}_n, X)$, $\wedge$ is also bounded on $A^2_\alpha(\mathbb{B}_n, X)$. Then by Theorem 6.2.3, there exists $g \in A^2_\alpha(\mathbb{B}_n, X^\star)$ such that

$$\wedge(f) = \int_{\mathbb{B}_n} \langle f(z), g(z)\rangle_{X,X^\star} \mathrm{d}\nu_\alpha(z), \tag{6.3.0.4}$$

for all $f \in A^2_\alpha(\mathbb{B}_n, X)$. Since $g \in A^2_\alpha(\mathbb{B}_n, X^\star)$, for any positive integer k, we have $D_k g \in A^2_{\alpha+k}(\mathbb{B}_n, X^\star)$. Applying Lemma 6.2.11 in (6.3.0.4), we obtain that

$$\wedge(f) = c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z)\rangle_{X,X^\star} (1-|z|^2)^k \mathrm{d}\nu_\alpha(z), \tag{6.3.0.5}$$

for all $f \in A^2_\alpha(\mathbb{B}_n, X)$. Now, we fix $x \in X$, $w \in \mathbb{B}_n$ and an integer $k > \gamma$. Let

$$f(z) = \frac{(1-|w|^2)^{k-\gamma}}{(1-\langle z, w\rangle)^{n+1+\alpha+k}} x, \quad z \in \mathbb{B}_n.$$

By Theorem 6.2.10, we have that $f \in A^2_\alpha(\mathbb{B}_n, X)$. Proposition 6.2.5 and (6.3.0.5), give us

$$\begin{aligned}
\wedge(f) &= c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z)\rangle_{X,X^\star} (1-|z|^2)^k \mathrm{d}\nu_\alpha(z) \\
&= c_k \int_{\mathbb{B}_n} \Big\langle \frac{(1-|w|^2)^{k-\gamma}}{(1-\langle z, w\rangle)^{n+1+\alpha+k}} x, D_k g(z)\Big\rangle_{X,X^\star} (1-|z|^2)^k \mathrm{d}\nu_\alpha(z) \\
&= \frac{c_\alpha c_k}{c_{\alpha+k}} (1-|w|^2)^{k-\gamma} \Big\langle x, \int_{\mathbb{B}_n} \frac{D_k g(z)}{(1-\langle w, z\rangle)^{n+1+\alpha+k}} \mathrm{d}\nu_{\alpha+k}(z)\Big\rangle_{X,X^\star} \\
&= \frac{c_\alpha c_k}{c_{\alpha+k}} (1-|w|^2)^{k-\gamma} \big\langle x, D_k g(w)\big\rangle_{X,X^\star}.
\end{aligned}$$

By Theorem 6.2.10, $f \in A^p_\alpha(\mathbb{B}_n, X)$ and $\|f\|_{p,\alpha,X} \lesssim \|x\|_X$. Since x is arbitrary, by duality, we have that

$$
\begin{aligned}
\|D_k g(w)\|_{X^\star} &= \sup_{\|x\|_X=1} |\langle x, D_k g(w)\rangle_{X,X^\star}| \\
&= \frac{c_{\alpha+k}}{c_\alpha c_k} \sup_{\|x\|_X=1} \frac{1}{(1-|w|^2)^{k-\gamma}} |\wedge(f)| \\
&\lesssim \sup_{\|x\|_X=1} \frac{1}{(1-|w|^2)^{k-\gamma}} \|\wedge\| \|f\|_{p,\alpha,X} \\
&\lesssim \sup_{\|x\|_X=1} \frac{\|\wedge\|}{(1-|w|^2)^{k-\gamma}} \|x\|_X \\
&\lesssim \frac{\|\wedge\|}{(1-|w|^2)^{k-\gamma}}.
\end{aligned}
$$

According to Corollary 6.2.14, we conclude that

$$
g \in \Gamma_\gamma(\mathbb{B}_n, X^\star) \quad \text{and} \quad \|g\|_{\Gamma_\gamma(\mathbb{B}_n, X^\star)} \lesssim \|\wedge\|,
$$

with $\gamma = (n+1+\alpha)\left(\frac{1}{p}-1\right)$. To finish the proof, it remains to show that (6.3.0.4) remains true for functions in $A^p_\alpha(\mathbb{B}_n, X)$ which is a direct consequence of the density in Corollary 6.2.2. $\square$

6.4 The Proofs of Theorem 6.1.4 and Corollary 6.1.5

In this section, we will give the proofs of Theorem 6.1.4 and Corollary 6.1.5.

6.4.1 Proof of Theorem 6.1.4

Proof. First assume that h_b extends to a bounded operator from $A^p_\alpha(\mathbb{B}_n, X)$ to $A^q_\alpha(\mathbb{B}_n, Y)$, with $q < 1$. Let $\|h_b\| := \|h_b\|_{A^p_\alpha(\mathbb{B}_n,X)\longrightarrow A^q_\alpha(\mathbb{B}_n,Y)}$. We want to show that $b \in \Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$. Since $h_b : A^p_\alpha(\mathbb{B}_n, X) \longrightarrow A^q_\alpha(\mathbb{B}_n, Y)$ is a bounded operator, we have by Theorem 6.1.3 that

$$|\langle h_b(f), g\rangle_{\alpha,Y}| \lesssim \|h_b\|\|f\|_{p,\alpha,X}\|g\|_{\Gamma_\beta(\mathbb{B}_n,Y^\star)},$$

for every $f \in A^p_\alpha(\mathbb{B}_n, X)$ and $g \in \Gamma_\beta(\mathbb{B}_n, Y^\star)$, with $\beta = (n+1+\alpha)\left(\frac{1}{q} - 1\right)$. Let $x \in X$, $y^\star \in Y^\star$, $w \in \mathbb{B}_n$ and an integer k such that $k > \gamma = (n + 1 + \alpha)\left(\frac{1}{p} - 1\right)$. Let $g(z) = y^\star$, and $f(z) = \dfrac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle z, w\rangle)^{n+1+\alpha+k}}x$. It is clear that $f \in H^\infty(\mathbb{B}_n, X)$ and $g \in \Gamma_\beta(\mathbb{B}_n, Y^\star)$, with $\|g\|_{\Gamma_\beta(\mathbb{B}_n,Y^\star)} = \|y^\star\|_{Y^\star}$. We also have by Theorem 6.2.10 that $f \in A^p_\alpha(\mathbb{B}_n, X)$, with $\|f\|_{p,\alpha,X} \lesssim \|x\|_X$. Hence

$$|\langle h_b(f), g\rangle_{\alpha,Y}| \lesssim \|h_b\|\|x\|_X\|y\star\|_{Y^\star}, \tag{6.4.1.1}$$

Applying Lemma 6.2.15 and the reproducing kernel property, we have that

$$\begin{aligned}
|\langle h_b(f), g\rangle_{\alpha,Y}| &= \left|\int_{\mathbb{B}_n} \langle b(z)\left(\overline{\frac{(1-|w|^2)^{k-\gamma}}{(1-\langle z,w\rangle)^{n+1+\alpha+k}}x}\right), y^\star\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z)\right| \\
&= (1-|w|^2)^{k-\gamma}\left|\int_{\mathbb{B}_n} \langle b(z)\left(\frac{\overline{x}}{(1-\langle w,z\rangle)^{n+1+\alpha+k}}\right), y^\star\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(\right. \\
&= (1-|w|^2)^{k-\gamma}\left|\int_{\mathbb{B}_n} \langle \frac{b(z)\,(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha+k}}, y^\star\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z)\right| \\
&= (1-|w|^2)^{k-\gamma}\left|\langle \int_{\mathbb{B}_n} \frac{b(z)\,(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha+k}} \mathrm{d}\nu_\alpha(z), y^\star\rangle_{Y,Y^\star}\right| \\
&= \frac{(1-|w|^2)^{k-\gamma}}{c_k}\left|\langle \int_{\mathbb{B}_n} L_k\left(\frac{b(z)\,(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha}}\right) \mathrm{d}\nu_\alpha(z), y^\star\rangle_{Y,Y^\star}\right| \\
&= \frac{(1-|w|^2)^{k-\gamma}}{c_k}\left|\langle L_k\left(\int_{\mathbb{B}_n} \frac{b(z)\,(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(z)\right), y^\star\rangle_{Y,Y^\star}\right| \\
&= \frac{(1-|w|^2)^{k-\gamma}}{c_k}\left|\langle L_k\left(b(w)\,(\overline{x})\right), y^\star\rangle_{Y,Y^\star}\right|.
\end{aligned}$$

Thus,

$$|\langle h_b(f), g\rangle_{\alpha,Y}| = \frac{(1-|w|^2)^{k-\gamma}}{c_k}\left|\langle L_k\left(b(w)\,(\overline{x})\right), y^\star\rangle_{Y,Y^\star}\right|. \tag{6.4.1.2}$$

From (6.4.1.1), (6.4.1.2) and the fact that $\|x\|_X = \|\overline{x}\|_{\overline{X}}$, we deduce that

$$(1-|w|^2)^{k-\gamma}\left|\langle L_k\left(b(w)\,(\overline{x})\right), y^\star\rangle_{Y,Y^\star}\right| \lesssim \|h_b\|\|\overline{x}\|_{\overline{X}}\|y^\star\|_{Y^\star}. \tag{6.4.1.3}$$

Since x and $y^\star$ are arbitrary, we get that

$$\sup_{w\in\mathbb{B}_n} (1-|w|^2)^{k-\gamma}\|L_k b(w)\|_{\mathcal{L}(\overline{X},Y^\star)} \lesssim \|h_b\|.$$

That is, $b\in\Gamma_\gamma(\mathbb{B}_n,\mathcal{L}(\overline{X},Y^\star))$ with $\|b\|_{\Gamma_\gamma(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))} \lesssim \|h_b\|$.

Conversely, assume that $b \in \Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ and let us prove that h_b extends to a bounded operator from $A^p_\alpha(\mathbb{B}_n, X)$ to $A^{1,\infty}_\alpha(\mathbb{B}_n, Y)$. Choose a positive integer $k > \gamma$, and let $f \in H^\infty(\mathbb{B}_n, X)$. Taking $g_z(w) = \dfrac{f(w)}{(1-\langle w, z\rangle)^{n+1+\alpha}}$, with $w \in \mathbb{B}_n$ and applying Lemma 6.2.16, Lemma 6.1.2 and the assumption we obtain

$$\begin{aligned}
\|h_b f(z)\|_Y &= \left\| \int_{\mathbb{B}_n} \frac{b(w)(\overline{f(w)})}{(1-\langle z, w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) \right\|_Y \\
&= c_k \left\| \int_{\mathbb{B}_n} L_k\left(b(w)\overline{g_z(w)}\right) \mathrm{d}\nu_{\alpha+k}(w) \right\|_Y \\
&= c_k \left\| \int_{\mathbb{B}_n} \frac{L_k\left(b(w)\overline{f(w)}\right)}{(1-\langle z, w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_{\alpha+k}(w) \right\|_Y \\
&\leq c_k \int_{\mathbb{B}_n} \left\| \frac{L_k\left(b(w)\overline{f(w)}\right)}{(1-\langle z, w\rangle)^{n+1+\alpha}} \right\|_Y \mathrm{d}\nu_{\alpha+k}(w) \\
&\leq \frac{c_k c_{\alpha+k}}{c_\alpha} \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{k-\gamma} \|L_k b(w)\|_{\mathcal{L}(\overline{\mathcal{X}}, \mathcal{Y})} \|\overline{f(w)}\|_{\overline{X}}}{|1-\langle z, w\rangle|^{n+1+\alpha}} (1-|w|^2)^\gamma \mathrm{d}\nu_\alpha \\
&\lesssim \|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))} \int_{\mathbb{B}_n} \frac{(1-|w|^2)^\gamma \|f(w)\|_X}{|1-\langle z, w\rangle|^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) \\
&= \|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))} P^+_\alpha g(z),
\end{aligned}$$

where the reproducing kernel is justified by (6.1.4.1) and

$$P^+_\alpha g(z) = \int_{\mathbb{B}_n} \frac{(1-|w|^2)^\gamma \|f(w)\|_X}{|1-\langle z, w\rangle|^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w)$$

is the positive Bergman operator of the positive function $g(z) = (1-|z|^2)^\gamma \|f(z)\|_X$. Now, let $\lambda > 0$. We have that

$$\nu_\alpha(\{z \in \mathbb{B}_n : \|h_b f(z)\|_Y > \lambda\}) \leq \nu_\alpha(\{z \in \mathbb{B}_n : c_k \|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))} P^+_\alpha g(z) > \lambda\}).$$

Since the positive Bergman operator $P_\alpha^+ : L^1_\alpha(\mathbb{B}_n) \longrightarrow L^{1,\infty}_\alpha(\mathbb{B}_n)$ is bounded (cf. e.g [85]), there exists a constant c such that

$$\begin{aligned}
\nu_\alpha(\{z \in \mathbb{B}_n : c_k \|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)} P_\alpha^+ g(z) > \lambda\}) &\leq \frac{c}{\frac{\lambda}{c_k \|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))}}} \|g\|_{L^1_\alpha(\mathbb{B}_n)} \\
&= \frac{cc_k}{\lambda} \|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))} \|g\|_{L^1_\alpha(\mathbb{B}_n)}.
\end{aligned}$$

Applying Lemma 6.2.7 to the function f, we get that

$$\begin{aligned}
\|g\|_{L^1_\alpha(\mathbb{B}_n)} &= \int_{\mathbb{B}_n} (1 - |z|^2)^\gamma \|f(z)\|_X \mathrm{d}\nu_\alpha(z) \\
&= \int_{\mathbb{B}_n} (1 - |z|^2)^{(\frac{1}{p} - 1)(n+1+\alpha)} \|f(z)\|_X \mathrm{d}\nu_\alpha(z) \\
&\leq \|f\|_{p,\alpha,X}.
\end{aligned}$$

It follows that

$$\lambda \nu_\alpha(\{z \in \mathbb{B}_n : \|h_b f(z)\|_Y > \lambda\}) \lesssim \|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))} \|f\|_{p,\alpha,X}$$

for all $\lambda > 0$. Therefore, h_b extends into a bounded operator from $A^p_\alpha(\mathbb{B}_n, X)$ to $A^{1,\infty}_\alpha(\mathbb{B}_n, Y)$ with

$$\|h_b\|_{A^p_\alpha(\mathbb{B}_n, X) \longrightarrow A^{1,\infty}_\alpha(\mathbb{B}_n, Y)} \lesssim \|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))}.$$

By density of $H^\infty(\mathbb{B}_n, X)$ on $A^p_\alpha(\mathbb{B}_n, X)$, the proof of the theorem is finished. □

6.4.2 Proof of Corollary 6.1.5

Proof. Just apply Lemma 6.2.9 and the second part of Theorem 6.1.4 to conclude. □

6.5 The Proof of Theorem 6.1.6

Proof. We first prove the sufficiency of the theorem. We assume that there exists a constant $C' > 0$ such that

$$\|N^k b(w)\|_{\mathcal{L}(\overline{X},Y)} \leq \frac{C'}{(1-|w|^2)^{k-\gamma}} \left(\log \frac{1}{1-|w|^2}\right)^{-1}.$$

Likewise by Corollary 6.2.14, we have that, there exists a constant $C > 0$ such that

$$\|L_k b(w)\|_{\mathcal{L}(\overline{X},Y)} \leq \frac{C}{(1-|w|^2)^{k-\gamma}} \left(\log \frac{1}{1-|w|^2}\right)^{-1}.$$

Applying Lemma 6.2.16 for any $f \in H^\infty(\mathbb{B}_n, X)$, we get

$$\int_{\mathbb{B}_n} \frac{b(w)(\overline{f(w)})}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) = c_k \int_{\mathbb{B}_n} \frac{L_k b(w)(\overline{f(w)})}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_{\alpha+k}(w).$$

Thus, by the assumption, Lemma 6.2.16 and Lemma 6.2.7 we have that

$$\begin{aligned}
\|h_b f\|_{A^1_\alpha(\mathbb{B}_n,Y)} &= \int_{\mathbb{B}_n} \left\| c_k \int_{\mathbb{B}_n} \frac{L_k b(w)(\overline{f(w)})}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_{\alpha+k}(w) \right\|_Y \mathrm{d}\nu_\alpha(z) \\
&\lesssim \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \left\| \frac{L_k b(w)(\overline{f(w)})}{(1-\langle z,w\rangle)^{n+1+\alpha}} \right\|_Y (1-|w|^2)^k \mathrm{d}\nu_\alpha(w) \mathrm{d}\nu_\alpha(z) \\
&\lesssim \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{\|L_k b(w)\|_{\mathcal{L}(\overline{X},Y)}}{|1-\langle z,w\rangle|^{n+1+\alpha}} \|\overline{f(w)}\|_{\overline{X}} (1-|w|^2)^k \mathrm{d}\nu_\alpha(w) \mathrm{d}\nu_\alpha(z) \\
&= \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{1}{|1-\langle z,w\rangle|^{n+1+\alpha}} \mathrm{d}\nu_\alpha(z) \right) \|L_k b(w)\|_{\mathcal{L}(\overline{X},Y)} \|\overline{f(w)}\|_{\overline{X}} (\\
&\lesssim \int_{\mathbb{B}_n} \left(\log \frac{1}{1-|w|^2}\right) \|\overline{f(w)}\|_{\overline{X}} \frac{(1-|w|^2)^k}{(1-|w|^2)^{k-\gamma}} \left(\log \frac{1}{1-|w|^2}\right)^{-1} \mathrm{d} \\
&= \int_{\mathbb{B}_n} \|f(w)\|_X (1-|w|^2)^\gamma \mathrm{d}\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} \|f(w)\|_X (1-|w|^2)^{(\frac{1}{p}-1)(n+1+\alpha)} \mathrm{d}\nu_\alpha(w) \\
&\lesssim \|f\|_{p,\alpha,X}.
\end{aligned}$$

Conversely, we assume that h_b extends into a bounded operator from $A^p_\alpha(\mathbb{B}_n, X)$ to $A^1_\alpha(\mathbb{B}_n, Y)$. Then for all $f \in H^\infty(\mathbb{B}_n, X)$ and $g \in \mathcal{B}(\mathbb{B}_n, Y^\star)$, we have

$$|\langle h_b(f), g\rangle_{\alpha,Y}| \leq \|h_b\| \|f\|_{p,\alpha,X} \|g\|_{\mathcal{B}(\mathbb{B}_n,Y^\star)}. \tag{6.5.0.1}$$

We choose the particular function $g(z) = y^\star$, with $y^\star \in Y^\star$. Applying Lemma 6.2.15, relation (6.5.0.1) becomes

$$\begin{aligned}\left|\int_{\mathbb{B}_n} \langle h_b f(z), y^\star\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z)\right| &= \left|\left\langle \int_{\mathbb{B}_n} b(z)\overline{f(z)}\mathrm{d}\nu_\alpha(z), y^\star\right\rangle_{Y,Y^\star}\right| \\ &\leq \|h_b\|\|f\|_{p,\alpha,X}\|y^\star\|_{Y^\star}.\end{aligned}$$

Thus

$$\left|\int_{\mathbb{B}_n} \langle b(z)\overline{f(z)}, y^\star\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z)\right| \leq \|h_b\|\|f\|_{p,\alpha,X}\|y^\star\|_{Y^\star} \tag{6.5.0.2}$$

for all $f \in H^\infty(\mathbb{B}_n, X)$ and $y^\star \in Y^\star$. Now, take $x \in X$, $y^\star \in Y^\star$, and an integer k such that $k > \gamma$. Fix $w \in \mathbb{B}_n$ and put

$$f(z) = \frac{(1-|w|^2)^{k-\gamma}}{(1-\langle z,w\rangle)^{n+1+\alpha+k}}x \quad ; \qquad g(z) = \log(1-\langle z,w\rangle)y^\star,$$

where log is the principal branch of the logarithm. Since $f \in H^\infty(\mathbb{B}_n, X)$ and $g \in \mathcal{B}(\mathbb{B}_n, Y^\star)$, by relation (6.5.0.1), we have that

$$|\langle h_b f, g\rangle|_{\alpha,Y} \leq \|h_b\|\|x\|_X\|y^\star\|_{Y^\star}. \tag{6.5.0.3}$$

Applying Lemma 6.2.15 for those particular vector-valued holomorphic functions f and g and using the fact that

$$\log(1-\langle w,z\rangle) = \log(1-|w|^2) + \log\left(\frac{1-\langle w,z\rangle}{1-|w|^2}\right),$$

we obtain

$$\begin{aligned}
\langle h_b f, g\rangle_{\alpha,Y} &= \int_{\mathbb{B}_n} \Big\langle b(z)\overline{\left(\frac{(1-|w|^2)^{k-\gamma}}{(1-\langle z,w\rangle)^{n+1+\alpha+k}}x\right)}, \log(1-\langle z,w\rangle)y^\star\Big\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z)\\
&= \Big\langle \int_{\mathbb{B}_n} b(z)\left[\frac{(1-|w|^2)^{k-\gamma}\log(1-\langle w,z\rangle)}{(1-\langle w,z\rangle)^{n+1+\alpha+k}}\overline{x}\right]\mathrm{d}\nu_\alpha(z), y^\star\Big\rangle_{Y,Y^\star}\\
&= \Big\langle \int_{\mathbb{B}_n} \frac{b(z)(\overline{x})(1-|w|^2)^{k-\gamma}\log(1-|w|^2)}{(1-\langle w,z\rangle)^{n+1+\alpha+k}}\mathrm{d}\nu_\alpha(z), y^\star\Big\rangle_{Y,Y^\star}\\
&+ \Big\langle \int_{\mathbb{B}_n} b(z)\left[\frac{(1-|w|^2)^{k-\gamma}}{(1-\langle w,z\rangle)^{n+1+\alpha+k}}\log\left(\frac{1-\langle w,z\rangle}{1-|w|^2}\right)\overline{x}\right]\mathrm{d}\nu_\alpha(z), y^\star\Big\rangle_{Y,
\end{aligned}$$

$$\begin{aligned}
&= \Big\langle (1-|w|^2)^{k-\gamma}\log(1-|w|^2)\int_{\mathbb{B}_n}\frac{b(z)(\overline{x})\mathrm{d}\nu_\alpha(z)}{(1-\langle w,z\rangle)^{n+1+\alpha+k}}, y^\star\Big\rangle_{Y,Y^\star}\\
&+ \Big\langle \int_{\mathbb{B}_n} b(z)\left(\overline{\frac{(1-|w|^2)^{k-\gamma}}{(1-\langle z,w\rangle)^{n+1+\alpha+k}}\log\left(\frac{1-\langle z,w\rangle}{1-|w|^2}\right)x}\right)\mathrm{d}\nu_\alpha(z), y^\star\Big\rangle
\end{aligned}$$

$$\begin{aligned}
&= (1-|w|^2)^{k-\gamma}\log(1-|w|^2)\Big\langle L_k\left(\int_{\mathbb{B}_n}\frac{b(z)(\overline{x})}{(1-\langle w,z\rangle)^{n+1+\alpha}}\mathrm{d}\nu_\alpha(z)\right),
\\
&+ \Big\langle \int_{\mathbb{B}_n} b(z)\left(\overline{f(z)\log\left(\frac{1-\langle z,w\rangle}{1-|w|^2}\right)}\right)\mathrm{d}\nu_\alpha(z), y^\star\Big\rangle_{Y,Y^\star}\\
&= (1-|w|^2)^{k-\gamma}\log(1-|w|^2)\langle L_k(b(w)(\overline{x})), y^\star\rangle_{Y,Y^\star} + \langle\int_{\mathbb{B}_n} b(z)(\overline{\varphi(z)
\end{aligned}$$

where $\varphi(z) = f(z)\log\left(\frac{1-\langle z,w\rangle}{1-|w|^2}\right)$. Therefore, we can write $\langle h_b f, g\rangle_{\alpha,Y} = I_1 + I_2$, with

$$I_1 = (1-|w|^2)^{k-\gamma}\log(1-|w|^2)\langle L_k(b(w)(\overline{x})), y^\star\rangle_{Y,Y^\star}$$

and

$$I_2 = \langle \int_{\mathbb{B}_n} b(z)(\overline{\varphi(z)})\mathrm{d}\nu_\alpha(z), y^\star \rangle_{Y,Y^\star}.$$

Applying Lemma 6.2.8 with $\delta = p$, and $\beta = p(k-\gamma)$, we obtain that

$$\begin{aligned}\|\varphi\|_{p,\alpha,X} &= \left(\int_{\mathbb{B}_n} \left|\log\left(\frac{1-\langle z,w\rangle}{1-|w|^2}\right)\right|^p \frac{(1-|w|^2)^{p(k-\gamma)}}{|1-\langle z,w\rangle|^{p(n+1+\alpha+k)}}\|x\|_X^p \mathrm{d}\nu_\alpha(z)\right)^{1/p} \\ &= \|x\|_X \left(\int_{\mathbb{B}_n} \left|\log\left(\frac{1-\langle z,w\rangle}{1-|w|^2}\right)\right|^p \frac{(1-|w|^2)^{p(k-\gamma)}}{|1-\langle z,w\rangle|^{n+1+\alpha+p(k-\gamma)}}\mathrm{d}\nu_\alpha(z)\right)^{1/p}\end{aligned}$$

According to the relation (6.5.0.2), we obtain the following estimation of I_2

$$|I_2| \leq \|h_b\|\|\varphi\|_{p,\alpha,X}\|y^\star\|_{Y^\star} \lesssim \|h_b\|\|x\|_X\|y^\star\|_{Y^\star}.$$

Since $I_1 = \langle h_b f, g\rangle_{\alpha,Y} - I_2$, by the relation (6.5.0.3) and the previous estimates on I_2, we have that

$$|I_1| \leq |\langle h_b f, g\rangle_{\alpha,Y}| + |I_2| \lesssim \|h_b\|\|x\|_X\|y^\star\|_{Y^\star}.$$

Since $x \in X$, $y^\star \in Y^\star$ are arbitrary and $\|x\|_X = \|\overline{x}\|_{\overline{X}}$, we get that

$$|I_1| = (1-|w|^2)^{k-\gamma}\log\left(\frac{1}{1-|w|^2}\right)|\langle L_k(b(w)(\overline{x})), y^\star\rangle_{Y,Y^\star}| \leq C\|h_b\|\|\overline{x}\|_{\overline{X}}\|y^\star\|_{Y^\star}.$$

Since $\overline{x} \in \overline{X}$ and $y^\star \in Y^\star$ are arbitrary, we deduce that :

$$\begin{aligned}\|L_k b(w)\|_{\mathcal{L}(\overline{X},Y)} &= \sup_{\|\overline{x}\|_{\overline{X}}=1, \|y^\star\|_{Y^\star}=1} |\langle L_k(b(w)(\overline{x})), y^\star\rangle_{Y,Y^\star}| \\ &\leq \frac{C}{(1-|w|^2)^{k-\gamma}}\left(\log\frac{1}{1-|w|^2}\right)^{-1}.\end{aligned}$$

The desired result follows at once using Corollary 6.2.14. □

6.6 Compactness of the little Hankel operator

In this section, we are going to characterize those symbols b for whch the little Hankel operator extends into a bounded compact oparator from $A^p_\alpha(\mathbb{B}_n, X)$ to

$A^q_\alpha(\mathbb{B}_n, Y)$, where $1 < p \leq q < \infty$ and X, Y are two reflexive complex Banach spaces.

6.6.1 Preliminaries notions

The proof of the following remark can be found in [94, Proposition 1.6.1]

Remark 6.6.1. Let $t \geq 0$. Then the operator $R^{\alpha,t}$ is the unique continuous linear operator on $\mathcal{H}(\mathbb{B}_n, X)$ satisfying

$$R^{\alpha,t}\left(\frac{x}{(1-\langle z,w\rangle)^{n+1+\alpha}}\right) = \frac{x}{(1-\langle z,w\rangle)^{n+1+\alpha+t}},$$

for every $z \in \mathbb{B}_n$ and $x \in X$.

We will use the operator $R^{\alpha,t}$, for $t > 0$, in the vector-valued Bergman space $A^1_\alpha(\mathbb{B}_n, X)$ as follows:

Proposition 6.6.2. *Let $t > 0$ and $f \in A^1_\alpha(\mathbb{B}_n, X)$. Then*

$$R^{\alpha,t} f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1-\langle z,w\rangle)^{n+1+\alpha+t}} \mathrm{d}\nu_\alpha(w),$$

for each $z \in \mathbb{B}_n$.

The proof of the following proposition is not quite different to the proof in [98, Proposition 1.15], but for sake of the completeness, we will recall the proof.

Proposition 6.6.3. *Suppose N is a positive integer and α is a real such that $n + \alpha$ is not a negative integer. Then $R^{\alpha,N}$ as an operator acting on $\mathcal{H}(\mathbb{B}_n, X)$ is a linear partial differential operator of order N with polynomial coefficients, that is*

$$R^{\alpha,N} f(z) = \sum_{m\in\mathbb{N}^n, |m|\leq N} p_m(z) \frac{\partial^{|m|} f}{\partial z^m}(z),$$

where each p_m is a polynomial.

Proof. Let $x \in X$ and $w \in \mathbb{B}_n$. By using the multi-nomial formula

$$\langle z, w\rangle^k = \sum_{|m|=k} \frac{k!}{m!} z^m \overline{w}^m,$$

it follows that

$$\begin{aligned}
\frac{x}{(1-\langle z,w\rangle)^{n+1+\alpha+N}} &= \frac{x(1-\langle z,w\rangle + \langle z,w\rangle)^N}{(1-\langle z,w\rangle)^{n+1+\alpha+N}} \\
&= \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} \frac{\langle z,w\rangle^k x\,(1-\langle z,w\rangle)^{N-k}}{(1-\langle z,w\rangle)^{n+1+\alpha+N}} \\
&= \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} \sum_{|m|=k} \frac{k!}{m!} z^m \frac{\overline{w}^m x}{(1-\langle z,w\rangle)^{n+1+\alpha+k}} \\
&= \sum_{k=0}^{N} \sum_{|m|=k} \frac{N!}{m!(N-k)!} z^m \frac{\overline{w}^m x}{(1-\langle z,w\rangle)^{n+1+\alpha+k}} \\
&= \sum_{k=0}^{N} \sum_{|m|=k} \frac{N!}{\prod_{j=0}^{k}(n+1+\alpha+j)m!(N-k)!} z^m \frac{\partial^k}{\partial z^m} \Bigg(\frac{\quad}{(1}
\end{aligned}$$

Therefore, there exists a constant c_{mk} such that

$$R^{\alpha,N}\left(\frac{x}{(1-\langle z,w\rangle)^{n+1+\alpha}}\right) = \sum_{k=0}^{N}\sum_{|m|=k} c_{mk} z^m \frac{\partial^k}{\partial z^m}\left(\frac{x}{(1-\langle z,w\rangle)^{n+1+\alpha}}\right).$$

Thus

$$R^{\alpha,N} = \sum_{k=0}^{N}\sum_{|m|=k} c_{mk} z^m \frac{\partial^k}{\partial z^m}.$$

$\square$

We will also need the following results whose proofs can be found in [94].

Lemma 6.6.4. *Let $t > 0$. Then*

$$\int_{\mathbb{B}_n} f(z)\overline{g(z)}\mathrm{d}\nu_\alpha(z) = \int_{\mathbb{B}_n} R^{\alpha,t} f(z)\overline{g(z)}\mathrm{d}\nu_{\alpha+t}(z),$$

for all $f \in A^1_\alpha(\mathbb{B}_n, X)$ and $g \in H^\infty(\mathbb{B}_n, \mathbb{C})$.

Lemma 6.6.5. *Let $t > 0$ and X a complex Banach space. Then*

$$\begin{aligned}
\int_{\mathbb{B}_n} \langle f(z), g(z)\rangle_{X,X^\star} \mathrm{d}\nu_\alpha(z) &= \int_{\mathbb{B}_n} \langle R^{\alpha,t} f(z), g(z)\rangle_{X,X^\star}\mathrm{d}\nu_{\alpha+t}(z) \\
&= \int_{\mathbb{B}_n} \langle f(z), R^{\alpha,t} g(z)\rangle_{X,X^\star}\mathrm{d}\nu_{\alpha+t}(z),
\end{aligned}$$

for every $f \in A^1_\alpha(\mathbb{B}_n, X)$ *and* $g \in H^\infty(\mathbb{B}_n, X^\star)$.

Corollary 6.6.6. *Suppose* $t > 0$ *and* $1 < p < \infty$. *If* $b \in A^{p'}_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, *where* p' *is the conjugate exponent of* p, *then the following equality holds*

$$\int_{\mathbb{B}_n} \langle b(z)\overline{f(z)}, g(z)\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z) = \int_{\mathbb{B}_n} \langle R^{\alpha,t} b(z)\overline{f(z)}, g(z)\rangle_{Y,Y^\star} \mathrm{d}\nu_{\alpha+t}(z)$$

for $f \in H^\infty(\mathbb{B}_n, X)$ *and* $g \in H^\infty(\mathbb{B}_n, Y^\star)$.

In the sequel, we will need to interchange the position of the summation symbol and the integral symbol in a particular situation. That is why we introduce this lemma.

Lemma 6.6.7. *Assume* $1 < t < \infty$. *Let* $b(z) = \sum_{\beta\in\mathbb{N}^n} \hat{b}(\beta) z^\beta \in A^t_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$. *Then*

$$\int_{\mathbb{B}_n} \langle b(z)\left(\overline{f(z)}\right), y_0^\star\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z) = \sum_{\beta\in\mathbb{N}^n} \int_{\mathbb{B}_n} z^\beta \langle \hat{b}(\beta)\left(\overline{f(z)}\right), y_0^\star\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z),$$

for every $f \in H^\infty(\mathbb{B}_n, X)$ *and* $y_0^\star \in Y^\star$ *with* $\|y_0^\star\|_{Y^\star} = 1$.

Proof. Since $b(z) = \sum_{\beta\in\mathbb{N}^n} \hat{b}(\beta) z^\beta \in A^t_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, we have that

$$\lim_{N\to\infty} \int_{\mathbb{B}_n} \left\| b(z) - \sum_{\beta\in\mathbb{N}^n, |\beta|\le N} \hat{b}(\beta) z^\beta \right\|^t_{\mathcal{L}(\overline{X},Y)} \mathrm{d}\nu_\alpha(z) = 0.$$

We have

$$\left| \int_{\mathbb{B}_n} \left\langle \left(b(z) - \sum_{\beta\in\mathbb{N}^n : |\beta|\le N} \hat{b}(\beta) z^\beta \right) (\overline{f(z)}), y_0^\star \right\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z) \right| \le$$

$$\int_{\mathbb{B}_n} \left\| b(z) - \sum_{\beta\in\mathbb{N}^n : |\beta|\le N} \hat{b}(\beta) z^\beta \right\|_{\mathcal{L}(\overline{X},Y)} \|\overline{f(z)}\|_{\overline{X}} \|y_0^\star\|_{Y^\star} \mathrm{d}\nu_\alpha(z) =$$

$$\int_{\mathbb{B}_n} \left\| b(z) - \sum_{\beta\in\mathbb{N}^n : |\beta|\le N} \hat{b}(\beta) z^\beta \right\|_{\mathcal{L}(\overline{X},Y)} \|f(z)\|_X \mathrm{d}\nu_\alpha(z) \lesssim$$

$$\int_{\mathbb{B}_n} \left\| b(z) - \sum_{\beta\in\mathbb{N}^n : |\beta|\le N} \hat{b}(\beta) z^\beta \right\|^t_{\mathcal{L}(\overline{X},Y)} \mathrm{d}\nu_\alpha(z) \longrightarrow 0$$

as $N \to \infty$. Therefore, we have that

$$\begin{aligned}
\int_{\mathbb{B}_n} \langle b(z)\left(\overline{f(z)}\right), y_0^\star\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z) &= \lim_{N\to\infty} \int_{\mathbb{B}_n} \left\langle \sum_{\beta\in\mathbb{N}^n:|\beta|\le N} \hat{b}(\beta) z^\beta \left(\overline{f(z)}\right), y_0^\star \right\rangle_{Y,Y^\star} \\
&= \lim_{N\to\infty} \int_{\mathbb{B}_n} \sum_{\beta\in\mathbb{N}^n:|\beta|\le N} \left\langle \hat{b}(\beta) z^\beta \left(\overline{f(z)}\right), y_0^\star \right\rangle_{Y,Y^\star} \\
&= \lim_{N\to\infty} \sum_{\beta\in\mathbb{N}^n:|\beta|\le N} \int_{\mathbb{B}_n} \left\langle \hat{b}(\beta) z^\beta \left(\overline{f(z)}\right), y_0^\star \right\rangle_{Y,Y^\star} \\
&= \sum_{\beta\in\mathbb{N}^n} \int_{\mathbb{B}_n} \left\langle \hat{b}(\beta) z^\beta \left(\overline{f(z)}\right), y_0^\star \right\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z).
\end{aligned}$$

□

In the following lemma, we compute the little Hankel operator when the operator-valued symbol is a monomial.

Lemma 6.6.8. *Suppose* $1 < p < \infty$ *and* $\gamma \in \mathbb{N}^n$. *If* $a_\gamma \in \mathcal{L}(\overline{X}, Y)$, *then for every* $f(z) = \sum_{\beta\in\mathbb{N}^n} c_\beta z^\beta \in A^p_\alpha(\mathbb{B}_n, X)$, *we have*

$$h_{a_\gamma z^\gamma} f(z) = \sum_{\beta\in\mathbb{N}^n, \beta\le\gamma} a_\gamma(\overline{c_\beta}) \frac{\gamma!\Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)!\Gamma(n+1+\alpha+|\gamma|)} z^{\gamma-\beta}.$$

Proof. Since

$$f(z) = \sum_{\beta\in\mathbb{N}^n} c_\beta z^\beta \in A^p_\alpha(\mathbb{B}_n, X),$$

and $p > 1$ by using [100, Corollary 4], it follows that

$$\int_{\mathbb{B}_n} \left\| \sum_{|\beta|\ge N+1} c_\beta z^\beta \right\|_X^p \mathrm{d}\nu_\alpha(z) \to 0 \quad \text{as} \quad N \to \infty. \tag{6.6.1.1}$$

Firstly, let us prove that

$$\int_{\mathbb{B}_n} \frac{\sum_{\beta\in\mathbb{N}^n} a_\gamma(\overline{c_\beta})\overline{w^\beta}}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) = \sum_{\beta\in\mathbb{N}^n} \int_{\mathbb{B}_n} \frac{a_\gamma(\overline{c_\beta})\overline{w^\beta}}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) \tag{6.6.1.2}$$

Let $N \in \mathbb{N}$. We have that

$$\int_{\mathbb{B}_n} \left\| \frac{\sum_{\beta\in\mathbb{N}^n} a_\gamma(\overline{c_\beta})\overline{w^\beta} - \sum_{|\beta|\le N} a_\gamma(\overline{c_\beta})\overline{w^\beta}}{(1-\langle z,w\rangle)^{n+1+\alpha}} \right\|_Y \mathrm{d}\nu_\alpha(w)$$

$$
\begin{aligned}
&= \int_{\mathbb{B}_n} \left\| \frac{\sum_{|\beta|\geq N+1} a_\gamma(\overline{c_\beta})\overline{w^\beta}}{(1-\langle z,w\rangle)^{n+1+\alpha}} \right\|_Y \mathrm{d}\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} \left\| \frac{a_\gamma\left(\sum_{|\beta|\geq N+1}(\overline{c_\beta})\overline{w^\beta}\right)}{(1-\langle z,w\rangle)^{n+1+\alpha}} \right\|_Y \mathrm{d}\nu_\alpha(w) \\
&\leq \frac{\|a_\gamma\|_{\mathcal{L}(\overline{X},Y)}}{(1-|z|)^{n+1+\alpha}} \int_{\mathbb{B}_n} \left\| \sum_{|\beta|\geq N+1} c_\beta w^\beta \right\|_X \mathrm{d}\nu_\alpha(w) \\
&\leq \frac{\|a_\gamma\|_{\mathcal{L}(\overline{X},Y)}}{(1-|z|)^{n+1+\alpha}} \int_{\mathbb{B}_n} \left\| \sum_{|\beta|\geq N+1} c_\beta w^\beta \right\|_X \mathrm{d}\nu_\alpha(w) \\
&\leq \frac{\|a_\gamma\|_{\mathcal{L}(\overline{X},Y)}}{(1-|z|)^{n+1+\alpha}} \left(\int_{\mathbb{B}_n} \left\| \sum_{|\beta|\geq N+1} c_\beta w^\beta \right\|_X^p \mathrm{d}\nu_\alpha(w) \right)^{1/p}.
\end{aligned}
$$

Therefore

$$
\left\| \int_{\mathbb{B}_n} \frac{\sum_{\beta\in\mathbb{N}^n} a_\gamma(\overline{c_\beta})\overline{w^\beta} - \sum_{|\beta|\leq N} a_\gamma(\overline{c_\beta})\overline{w^\beta}}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) \right\|_Y
$$

is less than or equal to

$$
\frac{\|a_\gamma\|_{\mathcal{L}(\overline{X},Y)}}{(1-|z|)^{n+1+\alpha}} \left(\int_{\mathbb{B}_n} \left\| \sum_{|\beta|\geq N+1} c_\beta w^\beta \right\|_X^p \mathrm{d}\nu_\alpha(w) \right)^{1/p}. \tag{6.6.1.3}
$$

By using (6.6.1.1) and (6.6.1.3), it follows that

$$
\left\| \int_{\mathbb{B}_n} \frac{\sum\limits_{\beta\in\mathbb{N}^n} a_\gamma(\overline{c_\beta})\overline{w^\beta} - \sum\limits_{|\beta|\leq N} a_\gamma(\overline{c_\beta})\overline{w^\beta}}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) \right\|_Y \to 0
$$

as $N\to\infty$, and so

$$
\begin{aligned}
\int_{\mathbb{B}_n} \frac{\sum\limits_{\beta\in\mathbb{N}^n} a_\gamma(\overline{c_\beta})\overline{w^\beta}}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) &= \lim_{N\to\infty} \int_{\mathbb{B}_n} \frac{\sum\limits_{|\beta|\leq N} a_\gamma(\overline{c_\beta})\overline{w^\beta}}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) \\
&= \lim_{N\to\infty} \sum_{|\beta|\leq N} \int_{\mathbb{B}_n} \frac{a_\gamma(\overline{c_\beta})\overline{w^\beta}}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) \\
&= \sum_{\beta\in\mathbb{N}^n} \int_{\mathbb{B}_n} \frac{a_\gamma(\overline{c_\beta})\overline{w^\beta}}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w),
\end{aligned}
$$

which is the desired result. Secondly, let us prove that

$$\int_{\mathbb{B}_n} \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z,w\rangle^k \mathrm{d}\nu_\alpha(w) = \sum_{k=0}^{\infty} \int_{\mathbb{B}_n} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z,w\rangle^k \mathrm{d}\nu_\alpha(w). \tag{6.6.1.4}$$

Let $N \in \mathbb{N}$. We have

$$\begin{aligned}
\left| \sum_{k=0}^{N} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z,w\rangle^k \right| &\leq \sum_{k=0}^{N} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} |z|^k \\
&\leq \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} |z|^k \\
&= \frac{1}{(1-|z|)^{n+1+\alpha}}.
\end{aligned}$$

Since

$$\int_{\mathbb{B}_n} \frac{1}{(1-|z|)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) = \frac{1}{(1-|z|)^{n+1+\alpha}},$$

by the dominated convergence theorem, we have that

$$\begin{aligned}
\sum_{k=0}^{\infty} \int_{\mathbb{B}_n} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z,w\rangle^k \mathrm{d}\nu_\alpha(w) &= \lim_{N\to\infty} \sum_{k=0}^{N} \int_{\mathbb{B}_n} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z,w\rangle \\
&= \lim_{N\to\infty} \int_{\mathbb{B}_n} \sum_{k=0}^{N} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z,w\rangle \\
&= \int_{\mathbb{B}_n} \lim_{N\to\infty} \sum_{k=0}^{N} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z,w\rangle \\
&= \int_{\mathbb{B}_n} \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z,w\rangle^k \mathrm{d}\nu_\alpha(
\end{aligned}$$

We are now ready to prove our lemma. For $f(z) = \sum_{\beta\in\mathbb{N}^n} c_\beta z^\beta \in A^p_\alpha(\mathbb{B}_n, X)$, by using the following multi-nomial formula ([98, (1.1)])

$$\langle z,w\rangle^k = \sum_{|m|=k} \frac{k!}{m!} z^m \overline{w^m}$$

and the following formula ([98, (1.23)])

$$\int_{\mathbb{B}_n} |z^m|^2 \nu_\alpha(z) = \frac{m!\Gamma(n+\alpha+1)}{\Gamma(n+|m|+\alpha+1)},$$

we get that

$$
\begin{aligned}
h_{a_\gamma z^\gamma} f(z) &= \int_{\mathbb{B}_n} \frac{a_\gamma w^\gamma \left(\overline{\sum\limits_{\beta\in\mathbb{N}^n} c_\beta w^\beta}\right)}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} \frac{w^\gamma \sum\limits_{\beta\in\mathbb{N}^n} a_\gamma(\overline{c_\beta})\overline{w^\beta}}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) \\
&= \sum_{\beta\in\mathbb{N}^n} \int_{\mathbb{B}_n} \frac{w^\gamma a_\gamma(\overline{c_\beta})\overline{w^\beta}}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) \quad \text{(by} \qquad (6.6.1.2)) \\
&= \sum_{\beta\in\mathbb{N}^n} a_\gamma(\overline{c_\beta}) \int_{\mathbb{B}_n} w^\gamma \overline{w^\beta} \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z,w\rangle^k \mathrm{d}\nu_\alpha(w) \\
&= \sum_{\beta\in\mathbb{N}^n} a_\gamma(\overline{c_\beta}) \sum_{k=0}^{\infty} \int_{\mathbb{B}_n} w^\gamma \overline{w^\beta} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z,w\rangle^k \mathrm{d}\nu_\alpha(w) \qquad \text{(by} \\
&= \sum_{\beta\in\mathbb{N}^n} a_\gamma(\overline{c_\beta}) \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \int_{\mathbb{B}_n} w^\gamma \overline{w^\beta} \sum_{|m|=k} \frac{k!}{m!} z^m \overline{w^m} \mathrm{d}\nu_\alpha(w) \\
&= \sum_{\beta\in\mathbb{N}^n} a_\gamma(\overline{c_\beta}) \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \sum_{|m|=k} \frac{k!}{m!} \int_{\mathbb{B}_n} w^\gamma \overline{w^\beta} z^m \overline{w^m} \mathrm{d}\nu_\alpha(w) \\
&= \sum_{\beta\in\mathbb{N}^n} a_\gamma(\overline{c_\beta}) \sum_{k=0}^{\infty} \sum_{|m|=k} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)m!} \int_{\mathbb{B}_n} w^\gamma z^m \overline{w^{m+\beta}} \mathrm{d}\nu_\alpha(w) \\
&= \sum_{\beta\in\mathbb{N}^n} a_\gamma(\overline{c_\beta}) \sum_{m\in\mathbb{N}^n} \frac{\Gamma(n+1+\alpha+|m|)}{\Gamma(n+1+\alpha)m!} z^m \int_{\mathbb{B}_n} w^\gamma \overline{w^{\beta+m}} \mathrm{d}\nu_\alpha(w) \\
&= \sum_{\beta\in\mathbb{N}^n, \beta\le\gamma} a_\gamma(\overline{c_\beta}) \frac{\Gamma(n+1+\alpha+|\gamma-\beta|)}{\Gamma(n+1+\alpha)(\gamma-\beta)!} z^{\gamma-\beta} \int_{\mathbb{B}_n} |z^\gamma|^2 \mathrm{d}\nu_\alpha(w) \\
&= \sum_{\beta\in\mathbb{N}^n, \beta\le\gamma} a_\gamma(\overline{c_\beta}) \frac{\Gamma(n+1+\alpha+|\gamma-\beta|)}{\Gamma(n+1+\alpha)(\gamma-\beta)!} \frac{\gamma!\Gamma(n+1+\alpha)}{\Gamma(n+1+\alpha+|\gamma|)} z^{\gamma-\beta} \\
&= \sum \quad (\overline{\quad}) \frac{\gamma!\Gamma(n+1+\alpha+|\gamma-\beta|)}{\quad} z^{\gamma-\beta}
\end{aligned}
$$

$\square$

The goal of the following lemma is to prove that the linear span of the vector-valued Bergman kernel $\dfrac{x^\star}{(1-\langle w,z\rangle)^{n+1+\alpha}}$, where $x^\star \in X^\star$ and $z, w \in \mathbb{B}_n$ form a dense subspace in the vector-valued Bergman space $A^{p'}_\alpha(\mathbb{B}_n, X^\star)$, with $1 < p < \infty$ and p' is the conjugate exponent of p.

Lemma 6.6.9. *Suppose that $1 < p < \infty$. For each $x^\star \in X^\star$ and $z \in \mathbb{B}_n$, let*

$$e_{z,x^\star}(w) = \frac{x^\star}{(1-\langle w,z\rangle)^{n+1+\alpha}}; \qquad w \in \mathbb{B}_n.$$

Then $e_{z,x^\star} \in A^{p'}_\alpha(\mathbb{B}_n, X^\star)$ and the subspace generated by $e_{z,x^\star}$ is dense in $A^{p'}_\alpha(\mathbb{B}_n, X^\star)$.

Proof. Let $\phi \in A^p_\alpha(\mathbb{B}_n, X)$ such that $\langle \phi, e_{z,x^\star}\rangle_{\alpha,X} = 0$ for all $z \in \mathbb{B}_n$ and $x^\star \in X^\star$. Let $f^\star \in A^{p'}_\alpha(\mathbb{B}_n, X^\star)$. According to the Hahn-Banach theorem, it suffices to prove that $\langle \phi, f^\star\rangle_{\alpha,X} = 0$. For all $z \in \mathbb{B}_n$ and $x^\star \in X^\star$, using Lemma 6.1.1 and the reproducing kernel formula, it follows that

$$\begin{aligned}
0 &= \langle \phi, e_{z,x^\star}\rangle_{\alpha,X} \\
&= \int_{\mathbb{B}_n} \langle \phi(w), e_{z,x^\star}(w)\rangle_{X,X^\star} \mathrm{d}\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} \langle \phi(w), \frac{x^\star}{(1-\langle w,z\rangle)^{n+1+\alpha}}\rangle_{X,X^\star} \mathrm{d}\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} \langle \frac{\phi(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}, x^\star\rangle_{X,X^\star} \mathrm{d}\nu_\alpha(w) \\
&= \langle \phi(z), x^\star\rangle_{X,X^\star}.
\end{aligned}$$

Therefore, for all $x^\star \in X^\star$, we have

$$\langle \phi(z), x^\star\rangle_{X,X^\star} = 0.$$

Thus $\phi(z) = 0$ for every $z \in \mathbb{B}_n$. It follows that for each $f^\star \in A^{p'}_\alpha(\mathbb{B}_n, X^\star)$, we have that

$$\langle \phi, f^\star \rangle_{\alpha,X} = \int_{\mathbb{B}_n} \langle \phi(z), f^\star(z) \rangle_{X,X^\star} \mathrm{d}\nu_\alpha(z) = 0.$$

□

In the proof of the following lemma, we use the fact that when X is a reflexive complex Banach space and $1 < p < \infty$, the dual of the vector-valued Bergman space $A^{p'}_\alpha(\mathbb{B}_n, X^\star)$ can be identified with $A^p_\alpha(\mathbb{B}_n, X)$, where p' is the conjugate exponent of p.

Lemma 6.6.10. *Suppose that $1 < p < \infty$, and X is a reflexive complex Banach space. Let $\{f_j\} \subset A^p_\alpha(\mathbb{B}_n, X)$ such that $f_j \to 0$ weakly in $A^p_\alpha(\mathbb{B}_n, X)$ as $j \to \infty$. Then for each $\beta \in \mathbb{N}^n$, we have that $\partial^\beta f_j(0) \to 0$ weakly in X as $j \to \infty$, where $\partial^\beta = \frac{\partial^{|\beta|}}{\partial z^\beta}$.*

Proof. Since for each $j \in \mathbb{N}$, $f_j \in A^p_\alpha(\mathbb{B}_n, X)$, using the reproducing kernel formula we have that

$$f_j(z) = \int_{\mathbb{B}_n} \frac{f_j(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w), \qquad z \in \mathbb{B}_n.$$

Differentiating both sides of the previous relation with respect to z, we obtain

$$\partial^\beta f_j(z) = C(n, \alpha, |\beta|) \int_{\mathbb{B}_n} \frac{f_j(w)\overline{w}^\beta}{(1 - \langle z, w \rangle)^{n+1+\alpha+|\beta|}} \mathrm{d}\nu_\alpha(w).$$

Therefore, we have

$$\partial^\beta f_j(0) = C(n, \alpha, |\beta|) \int_{\mathbb{B}_n} f_j(w)\overline{w}^\beta \mathrm{d}\nu_\alpha(w).$$

Now, let $x^\star \in X^\star$ and let us show that $\langle \partial^\beta f_j(0), x^\star \rangle_{X,X^\star} \to 0$ as $j \to \infty$. But we have that

$$\begin{aligned}
\langle \partial^\beta f_j(0), x^\star\rangle_{X,X^\star} &= C(n,\alpha,|\beta|)\left\langle \int_{\mathbb{B}_n} f_j(w)\overline{w}^\beta \mathrm{d}\nu_\alpha(w), x^\star\right\rangle_{X,X^\star}\\
&= \int_{\mathbb{B}_n} \langle f_j(w), x^\star w^\beta\rangle_{X,X^\star}\mathrm{d}\nu_\alpha(w)\\
&= \langle f_j, g\rangle_{\alpha,X} \to 0 \quad \text{as} \quad j\to\infty,
\end{aligned}$$

with $g(z) = x^\star z^\beta \in A_\alpha^{p'}(\mathbb{B}_n, X^\star)$. Thus, $\langle \partial^\beta f_j(0), x^\star\rangle_{X,X^\star} \to 0$ as $j\to\infty$. □

We recall that the symbol b used in the following lemma satisfies (6.1.4.1) and (6.1.6.2).

Lemma 6.6.11. *Suppose that X is a reflexive complex Banach space and k is a nonnegative integer. If the holomorphic mapping $z \mapsto b(z)$ maps $\mathbb{B}_n$ into $\mathcal{K}(\overline{X}, Y)$, then the holomorphic mapping $z \mapsto R^{\alpha,k}b(z)$ also maps $\mathbb{B}_n$ into $\mathcal{K}(\overline{X}, Y)$.*

Proof. Let $z \in \mathbb{B}_n$. Let $\{f_j\}$ a sequence of elements of X which converges weakly to 0 in X as j tends to infinity. Let us prove that $\lim_{j\to\infty} \|R^{\alpha,k}b(z)\overline{f_j}\|_Y = 0$. We know that the sequence $\{f_j\}$ is strongly bounded in X. Let $j \in \mathbb{N}$, by using (6.1.4.1) for $z = 0$, we get that the function $z \mapsto b(z)\overline{f_j} \in A_\alpha^1(\mathbb{B}_n, Y)$. By the reproducing kernel formula, it follows that

$$b(z)\overline{f_j} = \int_{\mathbb{B}_n} \frac{b(w)\overline{f_j}}{(1-\langle z,w\rangle)^{n+1+\alpha}}\mathrm{d}\nu_\alpha(w). \tag{6.6.1.5}$$

Applying the partial differential operator $R^{\alpha,k}$ to (6.6.1.5), we have

$$R^{\alpha,k}b(z)\overline{f_j} = \int_{\mathbb{B}_n} \frac{b(w)\overline{f_j}}{(1-\langle z,w\rangle)^{n+1+\alpha+k}}\mathrm{d}\nu_\alpha(w).$$

We also have

$$\begin{aligned}
\frac{\|b(w)\overline{f_j}\|_Y}{|1-\langle z,w\rangle|^{n+1+\alpha+k}} &\le \frac{\|b(w)\|_{\mathcal{L}(\overline{X},Y)}\|f_j\|_X}{(1-|z|)^{n+1+\alpha+k}}\\
&\le \frac{C(n+1+\alpha)}{(1-|z|)^{n+1+\alpha+k}}\|b(w)\|_{\mathcal{L}(\overline{X},Y)},
\end{aligned}$$

and

$$\int_{\mathbb{B}_n} \frac{C(n+1+\alpha)}{(1-|z|)^{n+1+\alpha+k}} \|b(w)\|_{\mathcal{L}(\overline{X},Y)} \mathrm{d}\nu_\alpha(w) < \infty.$$

Therefore, by applying the dominated convergence theorem, we have that

$$\begin{aligned} \limsup_{j\to\infty} \|R^{\alpha,k} b(z)\overline{f_j}\|_Y &\leq \limsup_{j\to\infty} \int_{\mathbb{B}_n} \frac{\|b(w)\overline{f_j}\|_Y}{|1-\langle z,w\rangle|^{n+1+\alpha+k}} \mathrm{d}\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \frac{\lim_{j\to\infty} \|b(w)\overline{f_j}\|_Y}{|1-\langle z,w\rangle|^{n+1+\alpha+k}} \mathrm{d}\nu_\alpha(w) = 0. \end{aligned}$$

Thu,s for each $z \in \mathbb{B}_n$

$$\lim_{j\to\infty} \|R^{\alpha,k} b(z)\overline{f_j}\|_Y = 0.$$

□

The following result will be also important in the sequel.

Lemma 6.6.12. *Suppose $\beta_0 \in \mathbb{N}^n$, $\{f_j\}$ a sequence of elements of X which converges weakly to 0 as j tends to infinity. For $z \in \mathbb{B}_n$, let $x_j(z) = z^{\beta_0} f_j$. Then $\{x_j\} \subset A^p_\alpha(\mathbb{B}_n, X)$ and $\{x_j\}$ converges weakly to 0 in $A^p_\alpha(\mathbb{B}_n, X)$.*

Proof. Let $j \in \mathbb{N}$. Since $f_j \to 0$ weakly in X as $j \to \infty$, it follows that $\{f_j\}$ is strongly bounded in X (see [92]). Let $\beta_0 \in \mathbb{N}^n$ and $x_j(z) = z^{\beta_0} f_j$. It is clear that $\{x_j\} \subset A^p_\alpha(\mathbb{B}_n, X)$. For every $g \in A^{p'}_\alpha(\mathbb{B}_n, X^\star)$, we have

$$\begin{aligned} \langle x_j, g\rangle_{\alpha,X} &= \int_{\mathbb{B}_n} \langle x_j(z), g(z)\rangle_{X,X^\star} \mathrm{d}\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \langle z^{\beta_0} f_j, g(z)\rangle_{X,X^\star} \mathrm{d}\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} z^{\beta_0} \langle f_j, g(z)\rangle_{X,X^\star} \mathrm{d}\nu_\alpha(z). \end{aligned}$$

Since

$$\begin{aligned} \left|z^{\beta_0} \langle f_j, g(z)\rangle_{X,X^\star}\right| &\leq |z^{\beta_0} \langle f_j, g(z)\rangle_{X,X^\star}| \\ &\leq \|f_j\|_X \|g(z)\|_{X^\star} \\ &\leq C\|g(z)\|_{X^\star}, \end{aligned}$$

and

$$\int_{\mathbb{B}_n} \|g(z)\|_{X^\star} \mathrm{d}\nu_\alpha(z) \leq \left(\int_{\mathbb{B}_n} \|g(z)\|_{X^\star}^{p'} \mathrm{d}\nu_\alpha(z)\right)^{1/p'} < \infty.$$

By using the dominated convergence theorem and the assumption, it follows that

$$\limsup_{j \longrightarrow \infty} \langle x_j, g\rangle_{\alpha,X} = \int_{\mathbb{B}_n} z^{\beta_0} \lim_{j\longrightarrow\infty} \langle f_j, g(z)\rangle_{X,X^\star} \mathrm{d}\nu_\alpha(z) = 0.$$

□

6.6.2 Boundedness of the little Hankel operator

The principal result here is that, the little Hankel operator with operator-valued symbol h_b is a bounded operator form $A^p_\alpha(\mathbb{B}_n, X)$ to $A^q_\alpha(\mathbb{B}_n, Y)$ with $1 < p \leq q < \infty$ if and only if the symbol b belongs to the generalized vector-valued Lipschitz space $\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, where

$$\gamma_0 = (n + 1 + \alpha)\left(\frac{1}{p} - \frac{1}{q}\right).$$

The result obtained generalize the Oliver's result [94, Theorem 4.2.2]. In the following lemma, we first prove that the definition of the generalized vector-valued Lipschitz space $\Lambda_\gamma(\mathbb{B}_n, X)$, with $\gamma \geq 0$ is independent of the integer k used.

Lemma 6.6.13. *Let $f \in \mathcal{H}(\mathbb{B}_n, X)$. The following conditions are equivalent:*

(a) There exists a nonnegative integer $k > \gamma$ such that

$$\sup_{z\in\mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_X < \infty.$$

(b) For every nonnegative integer $k > \gamma$ we have

$$\sup_{z\in\mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_X < \infty.$$

Proof. It is clear that $(b) \Rightarrow (a)$. So to complete the proof, we will prove that $(a) \Rightarrow (b)$. Suppose that there exists an integer $k > \gamma$ such that

$$c := \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_X < \infty.$$

We want to prove that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k+1-\gamma} \|R^{\alpha,k+1} f(z)\|_X < \infty.$$

Since $c < \infty$, then $f \in A^1_\alpha(\mathbb{B}_n, X)$. Indeed, by [94, Theorem 3.1.2], we have that

$$\begin{aligned}
\|f\|_{1,\alpha,X} &\simeq \int_{\mathbb{B}_n} (1 - |z|^2)^k \|R^{\alpha,k} f(z)\|_X \mathrm{d}\nu_\alpha(z) \\
&= \int_{\mathbb{B}_n} [(1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_X](1 - |z|^2)^\gamma \mathrm{d}\nu_\alpha(z) \\
&\lesssim c \int_{\mathbb{B}_n} (1 - |z|^2)^{\alpha+\gamma} \mathrm{d}\nu(z) \\
&< \infty.
\end{aligned}$$

By using Proposition 6.6.2, we have that

$$R^{\alpha,k+1} f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha+k+1}} \mathrm{d}\nu_\alpha(w).$$

Applyng Lemma 6.6.4, it follows that

$$R^{\alpha,k+1} f(z) = \int_{\mathbb{B}_n} \frac{R^{\alpha,k} f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha+k+1}} \mathrm{d}\nu_{\alpha+k}(w).$$

Thus,

$$\begin{aligned}
\|R^{\alpha,k+1} f(z)\|_X &\lesssim \int_{\mathbb{B}_n} \frac{[(1 - |w|^2)^{k-\gamma} \|R^{\alpha,k} f(w)\|_X](1 - |w|^2)^{\alpha+\gamma}}{|1 - \langle z, w \rangle|^{n+1+\alpha+\gamma+(k+1-\gamma)}} \mathrm{d}\nu(w) \\
&\lesssim \frac{c}{(1 - |z|^2)^{k+1-\gamma}}.
\end{aligned}$$

Therefore, we have that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k+1-\gamma} \|R^{\alpha,k+1} f(z)\|_X \lesssim c < \infty.$$

Also, if k is a nonnegative integer with $k > \gamma$ such that

$$c' := \sup_{z\in\mathbb{B}_n} (1-|z|^2)^{k+1-\gamma}\|R^{\alpha,k+1}f(z)\|_X < \infty,$$

then

$$\sup_{z\in\mathbb{B}_n} (1-|z|^2)^{k-\gamma}\|R^{\alpha,k}f(z)\|_X < \infty.$$

Applying Proposition 6.6.2 and Lemma 6.6.4 we have that

$$R^{\alpha,k}f(z) = \int_{\mathbb{B}_n} \frac{f(w)\mathrm{d}\nu_\alpha(w)}{(1-\langle z,w\rangle)^{n+1+\alpha+k}} = \int_{\mathbb{B}_n} \frac{R^{\alpha,k+1}f(w)\mathrm{d}\nu_{\alpha+k+1}(w)}{(1-\langle z,w\rangle)^{n+1+\alpha+k}},$$

where $z \in \mathbb{B}_n$. By using Theorem 6.2.10, it follows that

$$\begin{aligned}
\|R^{\alpha,k}f(z)\|_X &\lesssim \int_{\mathbb{B}_n} \frac{[(1-|w|^2)^{k+1-\gamma}\|R^{\alpha,k+1}f(w)\|_X](1-|w|^2)^{\alpha+\gamma}\mathrm{d}\nu(w)}{|1-\langle z,w\rangle|^{n+1+\alpha+k}} \\
&= c'\int_{\mathbb{B}_n} \frac{(1-|w|^2)^{\alpha+\gamma}\mathrm{d}\nu(w)}{|1-\langle z,w\rangle|^{n+1+\alpha+\gamma+(k-\gamma)}} \\
&\lesssim \frac{c'}{(1-|z|^2)^{k-\gamma}}.
\end{aligned}$$

Since $z \in \mathbb{B}_n$ is arbitrary, we obtain that

$$\sup_{z\in\mathbb{B}_n} (1-|z|^2)^{k-\gamma}\|R^{\alpha,k}f(z)\|_X \lesssim c' < \infty.$$

The proof of the lemma is complete. $\square$

Proposition 6.6.14. *Let $\gamma \geq 0$ and $f \in \Lambda_\gamma(\mathbb{B}_n, X)$. The following conditions are equivalent:*

(i) $f \in \Lambda_{\gamma,0}(\mathbb{B}_n, X)$.

(ii) $\lim_{s\to 1^-} \|f - f_s\|_{\Lambda_\gamma(\mathbb{B}_n,X)} = 0$, *where f_s is the dilation function defined for $z \in \mathbb{B}_n$ by $f_s(z) := f(sz)$.*

(iii) f belongs to the closure of $\mathcal{P}(\mathbb{B}_n, X)$, where $\mathcal{P}(\mathbb{B}_n, X)$ is the space of vector-valued holomorphic polynomials.

Proof. $(i) \Rightarrow (ii)$. Suppose that $\frac{1}{2} < r < s < 1$, and let $f_s(z) = f(sz), \quad z \in \mathbb{B}_n$.

By the definition, we have:

$$\begin{aligned}
\|f - f_s\|_{\Lambda_\gamma(\mathbb{B}_n, X)} &= \sup_{z\in\mathbb{B}_n} (1-|z|^2)^{k-\gamma} \|R^{\alpha,k}(f - f_s)(z)\|_X \\
&= \sup_{z\in\mathbb{B}_n} (1-|z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z) - (R^{\alpha,k} f_s)(z)\|_X \\
&= \sup_{z\in\mathbb{B}_n} (1-|z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z) - (R^{\alpha,k} f)(sz)\|_X \\
&= \sup_{z\in\mathbb{B}_n} (1-|z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z) - \chi_r(z)(R^{\alpha,k} f)(z) \\
&+ \chi_r(z)(R^{\alpha,k} f)(z) - (R^{\alpha,k} f)(sz)\|_X \\
&\le \sup_{z\in\mathbb{B}_n} (1-|z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z) - \chi_r(z)(R^{\alpha,k} f)(z)\|_X \\
&+ \sup_{z\in\mathbb{B}_n} (1-|z|^2)^{k-\gamma} \|\chi_r(z)(R^{\alpha,k} f)(z) - (R^{\alpha,k} f)(sz)\|_X,
\end{aligned}$$

where χ_r is the characteristic function of the set $\{|z| \le r\}$. We first have the following estimate:

$$\sup_{z\in\mathbb{B}_n} (1-|z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z) - \chi_r(z)(R^{\alpha,k} f)(z)\|_X \le$$

$$\begin{aligned}
& \sup_{|z|\le r} (1-|z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z) - \chi_r(z)(R^{\alpha,k} f)(z)\|_X \\
+ & \sup_{r<|z|<1} (1-|z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z) - \chi_r(z)(R^{\alpha,k} f)(z)\|_X \\
= & \sup_{r<|z|<1} (1-|z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z)\|_X \\
\le & \sup_{r^2<|z|<1} (1-|z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z)\|_X .
\end{aligned}$$

We secondly have the following estimate:

$$\sup_{z\in\mathbb{B}_n} (1-|z|^2)^{k-\gamma} \|\chi_r(z)(R^{\alpha,k} f)(z) - (R^{\alpha,k} f)(sz)\|_X \le$$

$$
\begin{aligned}
& \sup_{|z|\le r}(1-|z|^2)^{k-\gamma}\|\chi_r(z)(R^{\alpha,k}f)(z)-(R^{\alpha,k}f)(sz)\|_X \\
+ & \sup_{r<|z|<1}(1-|z|^2)^{k-\gamma}\|\chi_r(z)(R^{\alpha,k}f)(z)-(R^{\alpha,k}f)(sz)\|_X \\
= & \sup_{|z|\le r}(1-|z|^2)^{k-\gamma}\|(R^{\alpha,k}f)(z)-(R^{\alpha,k}f)(sz)\|_X \\
+ & \sup_{r<|z|<1}(1-|z|^2)^{k-\gamma}\|(R^{\alpha,k}f)(sz)\|_X.
\end{aligned}
$$

Using the change of variables $w = sz$, we then obtain

$$
\begin{aligned}
\sup_{r<|z|<1}(1-|z|^2)^{k-\gamma}\|(R^{\alpha,k}f)(sz)\|_X &= \sup_{rs<|w|<s}\left(1-\frac{|w|^2}{s^2}\right)^{k-\gamma}\|(R^{\alpha,k}f)(w)\|_X \\
&= \sup_{rs<|w|<s}\frac{1}{s^{2(k-\gamma)}}\left(s^2-|w|^2\right)^{k-\gamma}\|(R^{\alpha,k}f)(w)\|_X \\
&\le 2^{2(k-\gamma)}\sup_{r^2<|w|<1}(1-|w|^2)^{k-\gamma}\|(R^{\alpha,k}f)(w)\|_X.
\end{aligned}
$$

It follows by using the assumption that

$$
\begin{aligned}
\|f-f_s\|_{\Lambda_\gamma(\mathbb{B}_n,X)} \le\ & C_\gamma \sup_{r^2<|w|<1}(1-|w|^2)^{k-\gamma}\|(R^{\alpha,k}f)(w)\|_X \\
+ & \sup_{|z|\le r}(1-|z|^2)^{k-\gamma}\|(R^{\alpha,k}f)(z)-(R^{\alpha,k}f)(sz)\|_X,
\end{aligned}
$$

with $C_\gamma = 1+2^{2(k-\gamma)}$. Since $(R^{\alpha,k}f)(sz)\to(R^{\alpha,k}f)(z)$ in X uniformly on the compact set $\{|z|\le r\}$ as $s\to 1^-$, we have

$$
\lim_{s\to 1^-}\sup_{|z|\le r}(1-|z|^2)^{k-\gamma}\|(R^{\alpha,k}f)(z)-R^{\alpha,k}f(sz)\|_X = 0.
$$

It follows that

$$
\lim_{s\to 1^-}\|f-f_s\|_{\Lambda_\gamma(\mathbb{B}_n,X)} \le C_\gamma \limsup_{|w|\to 1^-}(1-|w|^2)^{k-\gamma}\|(R^{\alpha,k}f)(w)\|_X = 0.
$$

(ii) $\Rightarrow$ *(iii)*. Given $\epsilon > 0$, by the assumption, there exists $s_0\in(0,1)$ such that

$$
\|f-f_{s_0}\|_{\Lambda_\gamma(\mathbb{B}_n,X)} < \epsilon. \tag{6.6.2.1}
$$

Further note that $f_{s_0}\in\mathcal{H}(\frac{1}{s_0}\mathbb{B}_n,X)$ and $1<\frac{2}{1+s_0}<\frac{1}{s_0}$. From this, and by using Taylor's formula, it follows that for each $m\in\mathbb{N}$, there exists a $X-$valued polynomial p_m such that

$$\lim_{m\to\infty} \sup_{z\in\frac{2}{1+s_0}\overline{\mathbb{B}}_n} \|f_{s_0}(z) - p_m(z)\|_X = 0.$$

Therefore, there exists $m_0 \in \mathbb{N}$ such that

$$\sup_{z\in\frac{2}{1+s_0}\overline{\mathbb{B}}_n} \|f_{s_0}(z) - p_m(z)\|_X < \epsilon, \tag{6.6.2.2}$$

for $m \geq m_0$. By the Cauchy's inequality, there exists a constant $c_{s_0} > 0$ such that for each $i = 1, \cdots, n$ we have

$$\sup_{z\in\overline{\mathbb{B}}_n} \left\| \frac{\partial f_{s_0}}{\partial z_i} - \frac{\partial p_m}{\partial z_i} \right\|_X \leq c_{s_0} \sup_{z\in\frac{2}{1+s_0}\overline{\mathbb{B}}_n} \|f_{s_0}(z) - P_m(z)\|_X. \tag{6.6.2.3}$$

Suppose k is a nonnegative integer with $k > \gamma$. By using (6.6.2.3) and Theorem 6.6.3, there is a constant $c = c(s_0, n, \alpha, k)$ such that

$$\sup_{z\in\overline{\mathbb{B}}_n} \|(R^{\alpha,k} f_{s_0})(z) - (R^{\alpha,k} p_{m_0})(z)\|_X \leq c \sup_{z\in\frac{2}{1+s_0}\overline{\mathbb{B}}_n} \|f_{s_0}(z) - P_{m_0}(z)\|_X. \tag{6.6.2.4}$$

It follows by (6.6.2.4) and (6.6.2.2) that

$$\begin{aligned}
\sup_{z\in\mathbb{B}_n} (1-|z|^2)^{k-\gamma} \|R^{\alpha,k}(f_{s_0} - p_{m_0})(z)\|_X &\leq \sup_{z\in\mathbb{B}_n} \|(R^{\alpha,k} f_{s_0})(z) - (R^{\alpha,k} p_{m_0})(z)\| \\
&\leq c \sup_{z\in\frac{2}{1+s_0}\overline{\mathbb{B}}_n} \|f_{s_0}(z) - p_{m_0}(z)\|_X \\
&< c\epsilon.
\end{aligned}$$

Thus

$$\|f_{s_0} - p_{m_0}\|_{\Lambda_\gamma(\mathbb{B}_n, X)} < c\epsilon. \tag{6.6.2.5}$$

Using (6.6.2.1) and (6.6.2.5), it follows that

$$\begin{aligned}
\|f - p_{m_0}\|_{\Lambda_\gamma(\mathbb{B}_n,X)} &\leq \|f - f_{s_0}\|_{\Lambda_\gamma(\mathbb{B}_n,X)} + \|f_{s_0} - p_{m_0}\|_{\Lambda_\gamma(\mathbb{B}_n,X)} \\
&< \epsilon + c\epsilon = (1+c)\epsilon.
\end{aligned}$$

$(iii) \Rightarrow (i)$. Let f in the closure of the set of vector-valued polynomial $\mathcal{P}(\mathbb{B}_n, X)$, in $\Lambda_\gamma(\mathbb{B}_n, X)$. There exists a sequence of vector-valued polynomials $\{p_m\}$ in $\mathcal{P}(\mathbb{B}_n, X)$ such that

$$\lim_{m\to\infty} \|f - p_m\|_{\Lambda_\gamma(\mathbb{B}_n, X)} = 0. \tag{6.6.2.6}$$

Let us prove that for each $k > \gamma$,

$$\lim_{|z|\to 1^-} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z)\|_X = 0.$$

Let $k > \gamma$. We have that

$$\begin{aligned}
\|(R^{\alpha,k} f)(z)\|_X &\leq \|(R^{\alpha,k} f)(z) - (R^{\alpha,k} p_m)(z)\|_X + \|(R^{\alpha,k} p_m)(z)\|_X \\
&\leq \|(R^{\alpha,k} f)(z) - (R^{\alpha,k} p_m)(z)\|_X + \|R^{\alpha,k} p_m\|_{\infty,X},
\end{aligned}$$

where $\|R^{\alpha,k} p_m\|_{\infty,X} = \max_{z\in\mathbb{B}_n} \|(R^{\alpha,k} p_m)(z)\|_X$. It follows that for each $m \in \mathbb{N}$, we have

$$\begin{aligned}
(1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z)\|_X &\leq (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z) - (R^{\alpha,k} p_m)(z)\|_X \\
&\quad + (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} p_m\|_{\infty,X} \\
&\leq \|f - p_m\|_{\Lambda_\gamma(\mathbb{B}_n, X)} + (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} p_m\|_{\infty,X}.
\end{aligned}$$

Letting $|z| \to 1^-$, we obtain that

$$\limsup_{|z|\to 1^-} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_X \leq \|f - p_m\|_{\Lambda_\gamma(\mathbb{B}_n, X)},$$

for each $m \in \mathbb{N}$. Now, letting $m \to \infty$ on both sides of the previous inequality, it follows by (6.6.2.6) that

$$\limsup_{|z|\to 1^-} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_X = 0.$$

The proof of the proposition is complete. $\square$

Remark 6.6.15. One of the consequences of the previous result is that, given $\gamma \geq 0$, the generalized little vector-valued Lipschitz space $\Lambda_{\gamma,0}(\mathbb{B}_n, X)$ is a closed subspace of the generalized vector-valued Lipschitz space $\Lambda_\gamma(\mathbb{B}_n, X)$.

From now on, we choose $\gamma_0 = (n+1+\alpha)\left(\frac{1}{p} - \frac{1}{q}\right)$, with $1 < p \leq q < \infty$, and we consider the generalized vector-valued Lipschitz space $\Lambda_{\gamma_0}(\mathbb{B}_n, X)$.

Corollary 6.6.16. *Suppose* $1 \leq t < \infty$. *Then* $\Lambda_{\gamma_0}(\mathbb{B}_n, X) \subset A^t_\alpha(\mathbb{B}_n, X)$.

Proof. Let $k > \gamma_0$. Applying [94, Theorem 3.1.2], for $f \in \Lambda_{\gamma_0}(\mathbb{B}_n, X)$, we have that

$$\begin{aligned}
\|f\|^t_{t,\alpha,X} &\simeq \int_{\mathbb{B}_n} [(1-|z|^2)^k \|R^{\alpha,k} f(z)\|_X]^t \mathrm{d}\nu_\alpha(z) \\
&= \int_{\mathbb{B}_n} [(1-|z|^2)^{k-\gamma_0} \|R^{\alpha,k} f(z)\|_X]^t (1-|z|^2)^{\gamma_0 t} \mathrm{d}\nu_\alpha(z) \\
&\lesssim \|f\|_{\Lambda_{\gamma_0}(\mathbb{B}_n,X)} \int_{\mathbb{B}_n} (1-|z|^2)^{\alpha+\gamma_0 t} \mathrm{d}\nu(z) < \infty.
\end{aligned}$$

□

In what follows, we assume that X, Y are reflexives complex Banach spaces. We first introduce the following proposition which will be used in the proof of Theorem 6.1.8.

Proposition 6.6.17. *Suppose* $1 < p \leq q < \infty$, $0 \leq r < 1$ *and* $\gamma \in \mathbb{N}^n$. *If* $a_\gamma \in \mathcal{K}(\overline{X}, Y)$, *then the little Hankel operator* $h_{g_r^\gamma} : A^p_\alpha(\mathbb{B}_n, X) \to A^q_\alpha(\mathbb{B}_n, Y)$ *is a compact operator, where* $g_r^\gamma(z) = a_\gamma (rz)^\gamma$ *for every* $z \in \mathbb{B}_n$.

Proof. Let $\{f_j\}$ be a sequence in $A^p_\alpha(\mathbb{B}_n, X)$ such that $f_j \to 0$ weakly in $A^p_\alpha(\mathbb{B}_n, X)$ as j tends to infinity. We want to prove that $\lim_{j\to\infty} \|h_{g_r^\gamma} f_j\|_{q,\alpha,Y} = 0$. Let the Taylor expansion of f_j given by $f_j(z) = \sum_{\beta \in \mathbb{N}^n} c^j_\beta z^\beta \in A^p_\alpha(\mathbb{B}_n, X)$. Since $f_j \to 0$ weakly in $A^p_\alpha(\mathbb{B}_n, X)$, applying Lemma 6.6.10, using the fact that

$c_\beta^j = \partial^\beta f_j(0)/\beta!$, we have that for all $\beta \in \mathbb{N}^n$, $c_\beta^j \to 0$ weakly in X as $j \to \infty$. By Lemma 6.6.8, for every $z \in \mathbb{B}_n$, we have

$$h_{g_r^\gamma} f_j(z) = \sum_{\beta\in\mathbb{N}^n, \beta\le\gamma} a_\gamma(\overline{c_\beta^j}) \frac{\gamma!\Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)!\Gamma(n+1+\alpha+|\gamma|)} r^{|\gamma-\beta|} z^{\gamma-\beta}.$$

Therefore,

$$\begin{aligned}
\|h_{g_r^\gamma} f_j\|_{q,\alpha,Y} &= \left(\int_{\mathbb{B}_n} \left\| \sum_{\beta\in\mathbb{N}^n, \beta\le\gamma} a_\gamma(\overline{c_\beta^j}) \frac{\gamma!\Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)!\Gamma(n+1+\alpha+|\gamma|)} (rz)^{\gamma-\beta} \right\|_Y^q d\nu \right. \\
&\le \left(\int_{\mathbb{B}_n} \left(\sum_{\beta\in\mathbb{N}^n, \beta\le\gamma} \|a_\gamma(\overline{c_\beta^j})\|_Y \frac{\gamma!\Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)!\Gamma(n+1+\alpha+|\gamma|)} (r|z|)^{|\gamma-} \right.\right. \\
&\le \sum_{\beta\in\mathbb{N}^n, \beta\le\gamma} \left(\int_{\mathbb{B}_n} \left(\|a_\gamma(\overline{c_\beta^j})\|_Y \frac{\gamma!\Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)!\Gamma(n+1+\alpha+|\gamma|)} (r|z|)^{|\gamma-\beta} \right.\right. \\
&= \sum_{\beta\in\mathbb{N}^n, \beta\le\gamma} \|a_\gamma(\overline{c_\beta^j})\|_Y \frac{\gamma!\Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)!\Gamma(n+1+\alpha+|\gamma|)} \left(\int_{\mathbb{B}_n} (r|z|)^{|\gamma-\beta|q} \right. \\
&\lesssim \sum_{\beta\in\mathbb{N}^n, \beta\le\gamma} \frac{\gamma!\Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)!\Gamma(n+1+\alpha+|\gamma|)} \|a_\gamma(\overline{c_\beta^j})\|_Y,
\end{aligned}$$

where the third line above is justified by the Minkowsky's inequality for integrals. Thus,

$$\|h_{g_r^\gamma} f_j\|_{q,\alpha,Y} \lesssim \sum_{\beta\in\mathbb{N}^n, \beta\le\gamma} \frac{\gamma!\Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)!\Gamma(n+1+\alpha+|\gamma|)} \|a_\gamma(\overline{c_\beta^j})\|_Y. \tag{6.6.2.7}$$

Now, since $c_\beta^j \to 0$ weakly in X as $j \to \infty$, it is clear that $\overline{c_\beta^j} \to 0$ weakly in $\overline{X}$ as $j \to \infty$. By the assumption, we know that $a_\gamma \in \mathcal{K}(\overline{X}, Y)$. Since $\overline{c_\beta^j} \to 0$ weakly in $\overline{X}$ as $j \to \infty$, we have that $\|a_\gamma(\overline{c_\beta^j})\|_Y \to 0$ as $j \to \infty$. It follows that

$$\limsup_{j\to\infty} \|h_{g_r^\gamma} f_j\|_{q,\alpha,Y} \lesssim \sum_{\beta\in\mathbb{N}^n, \beta\le\gamma} \frac{\gamma!\Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)!\Gamma(n+1+\alpha+|\gamma|)} \lim_{j\to\infty} \|a_\gamma(\overline{c_\beta^j})\|_Y = 0.$$

$\square$

Let us state Oliver's result on the boundedness of the little Hankel operator with operator-valued symbol between vector-valued Bergman spaces.

Theorem 6.6.18. *Let $1 < p \leq q < \infty$. The little Hankel operator $h_b : A^p_\alpha(\mathbb{B}_n, X) \to A^q_\alpha(\mathbb{B}_n, Y)$ is a bounded operator if and only if $b \in \mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, where*

$$\gamma = 1 + (n+1+\alpha)\left(\frac{1}{q} - \frac{1}{p}\right).$$

Moreover

$$\|h_b\|_{A^p_\alpha(\mathbb{B}_n,X)\to A^q_\alpha(\mathbb{B}_n,Y)} \simeq \|b\|_{\mathcal{B}_\gamma(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}.$$

Remark 6.6.19. Suppose $1 < p < q < \infty$, and $\gamma = 1+(n+1+\alpha)\left(\frac{1}{q} - \frac{1}{p}\right)$. Then γ is not always positive. Indeed, since $1/q - 1/p \in (-1, 1)$, then $\gamma \in (-n-\alpha, n+2+\alpha)$. It follows that when $\gamma \in (-(n+\alpha), 0)$, the vector-valued γ-Bloch space $\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ is not interesting and does not make sense since the definition of the vector-valued γ-Bloch space introduced by Oliver only takes into account the case where $\gamma > 0$. In Theorem 6.1.7, we correct the problem by replacing the vector-valued γ-Bloch space with the generalized vector-valued Lipschitz space $\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, where $\gamma_0 = (n+1+\alpha)\left(\frac{1}{p} - \frac{1}{q}\right)$. Since $\gamma = 1 - \gamma_0$, we see that when $0 < \gamma_0 < 1$, we have that

$$\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)) = \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)).$$

In what follows, we give the proof of Theorem 6.1.7 which generalize the Theorem 6.6.18 and correct the mistake mentionned in Remark 6.6.19.

6.6.3 Proof of Theorem 6.1.7

Let us recall the statement of Theorem 6.1.7.

Theorem 6.6.20. *Suppose $1 < p \leq q < \infty$. The little Hankel operator h_b : $A^p_\alpha(\mathbb{B}_n, X)$*

$\to A^q_\alpha(\mathbb{B}_n, Y)$ is a bounded operator if and only if $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, where

$$\gamma_0 = (n+1+\alpha)\left(\frac{1}{p} - \frac{1}{q}\right).$$

Moreover, $\|h_b\|_{A^p_\alpha(\mathbb{B}_n,X)\to A^q_\alpha(\mathbb{B}_n,Y)} \simeq \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}$.

Proof. Let p' and q' such that $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. We first assume that h_b is a bounded operator from $A^p_\alpha(\mathbb{B}_n, X)$ to $A^q_\alpha(\mathbb{B}_n, Y)$ with norm $\|h_b\| = \|h_b\|_{A^p_\alpha(\mathbb{B}_n,X)\to A^q_\alpha(\mathbb{B}_n,Y)}$. Let $x \in X$ and $k > (n+1+\alpha)/p$. Let $z \in \mathbb{B}_n$ and put

$$f(w) = \frac{x}{(1-\langle w, z\rangle)^k}, \qquad w \in \mathbb{B}_n.$$

Since $k > (n+1+\alpha)/p$, by Theorem 6.2.10, we have that $f \in A^p_\alpha(\mathbb{B}_n, X)$ and

$$\|f\|_{p,\alpha,X} \lesssim \frac{\|x\|_X}{(1-|z|^2)^{k-(n+1+\alpha)/p}}.$$

By [94, Proposition 2.1.3], we have that

$$\begin{aligned} h_b f(z) &= \int_{\mathbb{B}_n} \frac{b(w)(\overline{f(w)})}{(1-\langle z, w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \frac{b(w)(\overline{x})}{(1-\langle z, w\rangle)^{n+1+\alpha+k}} \mathrm{d}\nu_\alpha(w) \\ &= R^{\alpha,k} b(z)(\overline{x}). \end{aligned}$$

It follows by Theorem 6.2.6 that

$$\begin{aligned} \|R^{\alpha,k} b(z)(\overline{x})\|_Y &= \|h_b f(z)\|_Y \\ &\leq \frac{\|h_b f\|_{q,\alpha,Y}}{(1-|z|^2)^{(n+1+\alpha)/q}} \\ &\leq \frac{\|h_b\| \|f\|_{p,\alpha,X}}{(1-|z|^2)^{(n+1+\alpha)/q}} \\ &\lesssim \frac{\|h_b\| \|x\|_X}{(1-|z|^2)^{k+(n+1+\alpha)(1/q-1/p)}} \\ &= \frac{\|h_b\| \|x\|_X}{(1-|z|^2)^{k-\gamma_0}}. \end{aligned}$$

Since $x \in X$ is arbitrary and $\|x\|_X = \|\overline{x}\|_{\overline{X}}$ we get that

$$\|R^{\alpha,k}b(z)\|_{\mathcal{L}(\overline{X},Y)} \lesssim \frac{\|h_b\|}{(1-|z|^2)^{k-\gamma_0}}.$$

Thus

$$\sup_{z\in\mathbb{B}_n}(1-|z|^2)^{k-\gamma_0}\|R^{\alpha,k}b(z)\|_{\mathcal{L}(\overline{X},Y)} \lesssim \|h_b\|.$$

By Lemma 6.6.13 this means that $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ and $\|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))} \lesssim \|h_b\|$.

Conversely, assume that $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$. Let $f \in A^p_\alpha(\mathbb{B}_n, X)$, $g \in A^{q'}_\alpha(\mathbb{B}_n, Y^\star)$ and $k > \gamma_0$. By Corollary 6.6.16, we have that

$$b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)) \subset A^{p'}_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)),$$

so by [94, Lemma 4.1.1], Corollary 6.6.6, and Lemma 6.6.13 it follows that

$$\begin{aligned}
|\langle h_b f, g\rangle_{\alpha,Y}| &= \left|\int_{\mathbb{B}_n}\langle b(z)\overline{f(z)}, g(z)\rangle_Y \mathrm{d}\nu_\alpha(z)\right| \\
&= \left|\int_{\mathbb{B}_n}\langle R^{\alpha,k+1}b(z)\overline{f(z)}, g(z)\rangle_Y d\nu_{\alpha+k+1}(z)\right| \\
&\lesssim \int_{\mathbb{B}_n}\|R^{\alpha,k+1}b(z)\|_{\mathcal{L}(\overline{X},Y)}\|\overline{f(z)}\|_{\overline{X}}\|g(z)\|_{Y^\star}(1-|z|^2)^{k+1+\alpha}\mathrm{d}\nu(z) \\
&\lesssim \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}\int_{\mathbb{B}_n}\|f(z)\|_X\|g(z)\|_{Y^\star}(1-|z|^2)^{\alpha+\gamma_0}\mathrm{d}\nu(z).
\end{aligned}$$

By Hölder's inequality the last integral is less than or equal to

$$\left(\int_{\mathbb{B}_n}\|f(z)\|^q_X(1-|z|^2)^{\alpha+q\gamma_0}\mathrm{d}\nu(z)\right)^{1/q}\left(\int_{\mathbb{B}_n}\|g(z)\|^{q'}_{Y^\star}(1-|z|^2)^{\alpha}\mathrm{d}\nu(z)\right)^{1/q'}.$$

For $q = p$, we have $\gamma_0 = 0$ and thus

$$|\langle h_b f, g\rangle_{\alpha,Y}| \lesssim \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}\|f\|_{p,\alpha,X}\|g\|_{p',\alpha,Y}.$$

For $q - p > 0$, using Theorem 6.2.6, we have

$$\|f(z)\|^q_X = \|f(z)\|^p_X\|f(z)\|^{q-p}_X \leq \frac{\|f(z)\|^p_X\|f\|^{q-p}_{p,\alpha,X}}{(1-|z|^2)^{(q-p)(n+1+\alpha)/p}} = \frac{\|f(z)\|^p_X\|f\|^{q-p}_{p,\alpha,X}}{(1-|z|^2)^{q\gamma_0}}.$$

It follows that

$$\left(\int_{\mathbb{B}_n} \|f(z)\|_X^q (1-|z|^2)^{\alpha+q\gamma_0} \mathrm{d}\nu(z)\right)^{1/q} \leq \|f\|_{p,\alpha,X}^{1-p/q} \left(\int_{\mathbb{B}_n} \|f(z)\|_X^p \frac{(1-|z|^2)^{\alpha+q\gamma_0}}{(1-|z|^2)^{q\gamma_0}} \right.$$
$$= \|f\|_{p,\alpha,X}.$$

Therefore, by duality, we obtain that

$$\|h_b\|_{A^p_\alpha(\mathbb{B}_n,X)\to A^q_\alpha(\mathbb{B}_n,Y)} = \sup_{\|f\|_{p,\alpha,X}=1;\|g\|_{q',\alpha,Y^\star}=1} |\langle h_b f, g\rangle_{\alpha,Y}| \lesssim \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}.$$

□

6.6.4 Proof of Theorem 6.1.8

We are now ready to give the proof of the main result in this section that is Theorem 6.1.8 that we recall here.

Theorem 6.6.21. *Let X and Y be two reflexive complex Banach spaces. Suppose that $1 < p \leq q < \infty$, and $\alpha > -1$ The little Hankel operator h_b : $A^p_\alpha(\mathbb{B}_n, X) \longrightarrow A^q_\alpha(\mathbb{B}_n, Y)$ is a compact operator if and only if*

$$b \in \Lambda_{\gamma_0,0}(\mathbb{B}_n, \mathcal{K}(\overline{X}, Y)),$$

where $\Lambda_{\gamma_0,0}(\mathbb{B}_n, \mathcal{K}(\overline{X}, Y))$ denotes the generalized little vector-valued Lipschitz space and $\gamma_0 = (n+1+\alpha)\left(\frac{1}{p}-\frac{1}{q}\right)$, see (6.1.3.3).

Proof. First assume that $b \in \Lambda_{\gamma_0,0}(\mathbb{B}_n, \mathcal{K}(\overline{X}, Y))$ and denote by $b_r(z) := b(rz)$ with $z \in \mathbb{B}_n$ and $0 < r < 1$. Since $b \in \Lambda_{\gamma_0,0}(\mathbb{B}_n, \mathcal{K}(\overline{X}, Y))$, by Theorem 6.1.7, we have that

$$\|h_b\|_{A^p_\alpha(\mathbb{B}_n,X)\to A^q_\alpha(\mathbb{B}_n,Y)} \lesssim \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}.$$

Therefore, we have

$$\|h_b - h_{b_r}\|_{A^p_\alpha(\mathbb{B}_n,X)\to A^q_\alpha(\mathbb{B}_n,Y)} \lesssim \|b - b_r\|_{\Lambda_{\gamma_0}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}.$$

By using Proposition 6.6.14, we have that

$$\lim_{r\to 1^-} \|b - b_r\|_{\Lambda_{\gamma_0}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))} = 0,$$

so to prove that h_b is a compact operator, it suffices to prove that h_{b_r} is a compact operator. Since b_r is analytic on a neighbourhood of $\overline{\mathbb{B}}_n$, it can be approximated by its Taylor polynomial in the Bloch norm. Thus,

$$\lim_{N\to\infty} \|b_r - P_{N,r}\|_{\Lambda_{\gamma_0}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))} = 0, \tag{6.6.4.1}$$

with $P_{N,r}(z) = \sum_{\beta\in\mathbb{N}^n, |\beta|\leq N} \hat{b}(\beta) r^{|\beta|} z^\beta$, where $\hat{b}(\beta) \in \mathcal{K}(\overline{X}, Y)$ are the Taylor coefficients of b. We also have by Theorem 6.1.7 that

$$\|h_{b_r} - h_{P_{N,r}}\|_{A^p_\alpha(\mathbb{B}_n,X)\to A^q_\alpha(\mathbb{B}_n,Y)} \lesssim \|b_r - P_{N,r}\|_{\Lambda_{\gamma_0}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}.$$

So by (6.6.4.1), to prove that h_{b_r} is a compact operator, it is enough to prove that $h_{P_{N,r}}$ is a compact operator. Since $P_{N,r}$ is a polynomial, it is enough to do the proof for monomials of the form $\hat{b}(\beta) r^{|\beta|} z^\beta$, with $\beta \in \mathbb{N}^n$, $z \in \mathbb{B}_n$ and $\hat{b}(\beta) \in K(\overline{X}, Y)$. Thus, according to Proposition 6.6.17, the proof of this part is complete.

Conversely, for the "only if part", let us assume that

$$h_b : A^p_\alpha(\mathbb{B}_n, X) \longrightarrow A^q_\alpha(\mathbb{B}_n, Y)$$

is a compact operator. Since h_b is compact, h_b is then bounded and Theorem 6.1.7 yields

$$b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)).$$

We shall first prove that the Taylor coefficients $\hat{b}(\beta)$, $\beta \in \mathbb{N}^n$ of b belongs to $\mathcal{K}(\overline{X}, Y)$. Let $\{f_j\} \subset X$ such that $f_j \longrightarrow 0$ weakly in X as $j \longrightarrow \infty$, fix $\beta_0 \in \mathbb{N}^n$, and let $x_j(z) = z^{\beta_0} f_j$. By Lemma 6.6.12, we have $\{x_j\} \subset A^p_\alpha(\mathbb{B}_n, X)$ and $\{x_j\}$

converges weakly to 0 in $A^p_\alpha(\mathbb{B}_n, X)$. Since

$$\|\hat{b}(\beta_0)\overline{f_j}\|_Y = \sup_{\|y^\star\|_{Y^\star}=1} |\langle \hat{b}(\beta_0)\overline{f_j}, y^\star\rangle_{Y,Y^\star}|$$

and Y is reflexive, by the Kakutani's theorem [92, Theorem 3.17] there exists $y_j^\star \in Y^\star$ with $\|y_j^\star\|_{Y^\star} = 1$ such that

$$\|\hat{b}(\beta_0)\overline{f_j}\|_Y = |\langle \hat{b}(\beta_0)\overline{f_j}, y_j^\star\rangle_{Y,Y^\star}|.$$

But $y_j^\star \in A^{p'}_\alpha(\mathbb{B}_n, Y^\star)$. By Lemma 6.2.15, we have

$$\begin{aligned}
|\langle h_b x_j, y_j^\star\rangle_{\alpha,Y}| &= \left|\int_{\mathbb{B}_n} \langle b(z)\overline{x_k}(z), y_j^\star\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z)\right| \\
&= \left|\int_{\mathbb{B}_n} \overline{z}^{\beta_0}\langle \sum_{\beta\in\mathbb{N}^n} z^\beta \hat{b}(\beta)\overline{f_j}, y_j^\star\rangle_{Y,Y^\star} \mathrm{d}\nu_\alpha(z)\right| \\
&= \left|\sum_{\beta\in\mathbb{N}^n} \langle \hat{b}(\beta)\overline{f_j}, y_j^\star\rangle_{Y,Y^\star} \int_{\mathbb{B}_n} z^\beta \overline{z}^{\beta_0} \mathrm{d}\nu_\alpha(z)\right| \\
&= |\langle \hat{b}(\beta_0)\overline{f_j}, y_j^\star\rangle_{Y,Y^\star}| \int_{\mathbb{B}_n} |z^{\beta_0}|^2 \mathrm{d}\nu_\alpha(z) \\
&= \frac{\beta_0!\Gamma(n+\alpha+1)}{\Gamma(n+|\beta_0|+\alpha+1)} |\langle \hat{b}(\beta_0)\overline{f_j}, y_j^\star\rangle_{Y,Y^\star}| \\
&= \frac{\beta_0!\Gamma(n+\alpha+1)}{\Gamma(n+|\beta_0|+\alpha+1)} \|\hat{b}(\beta_0)\overline{f_j}\|_Y,
\end{aligned}$$

where Fubini's theorem is justified by Lemma 6.6.7 with $\{x_j\} \subset H^\infty(\mathbb{B}_n, X)$. Since h_b is compact and $\{x_j\}$ converges weakly to 0 as j tends to infinity, we have that $\{h_b x_j\}$ converges strongly to 0 as j tends to infinity, therefore one gets that

$$\lim_{j\to\infty} \langle h_b x_j, y_j^\star\rangle_{\alpha,Y} = 0.$$

Thus

$$\lim_{j\to\infty} \frac{\beta_0!\Gamma(n+\alpha+1)}{\Gamma(n+|\beta_0|+\alpha+1)} \|\hat{b}(\beta_0)\overline{f_j}\|_Y = 0.$$

We then obtain

$$\lim_{j\to\infty} \|\hat{b}(\beta_0)\overline{f_j}\|_Y = 0.$$

In fact, we have shown that $\hat{b}(\beta_0)$ belongs to $\mathcal{K}(\overline{X}, Y)$ and as β_0 is arbitrary, this holds for all $\beta \in \mathbb{N}^n$. Let $1 < t < \infty$. Since $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, we have that $b \in A^t_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ and

$$\lim_{N\to\infty} \int_{\mathbb{B}_n} \|b(w) - \sum_{|\beta|\le N} \hat{b}(\beta)w^\beta\|^t_{\mathcal{L}(\overline{X},Y))} \mathrm{d}\nu_\alpha(w) = 0.$$

Let $z \in \mathbb{B}_n$. There exists a constant $C_z > 0$ such that

$$\|b(z) - \sum_{|\beta|\le N} \hat{b}(\beta)z^\beta\|^t_{\mathcal{L}(\overline{X},Y))} \le C_z \int_{\mathbb{B}_n} \|b(w) - \sum_{|\beta|\le N} \hat{b}(\beta)w^\beta\|^t_{\mathcal{L}(\overline{X},Y))} \mathrm{d}\nu_\alpha(w).$$

Thus,

$$\lim_{N\to\infty} \|b(z) - \sum_{|\beta|\le N} \hat{b}(\beta)z^\beta\|_{\mathcal{L}(\overline{X},Y))} = 0.$$

Since $z \in \mathbb{B}_n$ is arbitrary, we deduce that $b(z) \in \mathcal{K}(\overline{X}, Y)$, for each $z \in \mathbb{B}_n$. Let $x \in X$ and $y^\star \in Y^\star$. Since $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, then the mapping $z \mapsto \langle b(z)\overline{x}, y^\star\rangle_{Y,Y^\star}$ belongs to $A^1_\alpha(\mathbb{B}_n, \mathbb{C})$. By using the reproducing kernel formula, it follows that

$$\langle b(z)\overline{x}, y^\star\rangle_{Y,Y^\star} = \int_{\mathbb{B}_n} \frac{\langle b(w)\overline{x}, y^\star\rangle_{Y,Y^\star}}{(1-\langle z,w\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(w). \tag{6.6.4.2}$$

Let $k > \gamma_0$. Applying the operator $R^{\alpha,k}$ in (6.6.4.2), we obtain that

$$\langle R^{\alpha,k} b(z)\overline{x}, y^\star\rangle_{Y,Y^\star} = \int_{\mathbb{B}_n} \frac{\langle b(w)\overline{x}, y^\star\rangle_{Y,Y^\star}}{(1-\langle z,w\rangle)^{n+1+\alpha+k}} \mathrm{d}\nu_\alpha(w). \tag{6.6.4.3}$$

Let $z \in \mathbb{B}_n$. Since $\|R^{\alpha,k}b(z)\|_{\mathcal{L}(\overline{X},Y)} = \sup_{\|x\|_X=1} \|R^{\alpha,k}b(z)(\overline{x})\|_Y$, and by Lemma 6.6.11, the operator $R^{\alpha,k}b(z)$ is compact. So there exists $x_0(z) \in X$ with $\|x_0(z)\|_X = 1$ and

$$\|R^{\alpha,k}b(z)\|_{\mathcal{L}(\overline{X},Y)} = \|R^{\alpha,k}b(z)\overline{x_0(z)}\|_Y.$$

Also

$$\|R^{\alpha,k}b(z)\overline{x_0(z)}\|_Y = \sup_{\|y^\star\|_{Y^\star}=1} |\langle R^{\alpha,k}b(z)\overline{x_0(z)}, y^\star\rangle_{Y,Y^\star}|.$$

Since Y is reflexive, it follows by the Kakutani's theorem [92, Theorem 3.17] that there exists $y_0^\star(z) \in Y^\star$ with $\|y_0^\star(z)\|_{Y^\star} = 1$ such that

$$\|R^{\alpha,k}b(z)\|_{\mathcal{L}(\overline{X},Y)} = \|R^{\alpha,k}b(z)(\overline{x_0(z)})\|_Y = |\langle R^{\alpha,k}b(z)\overline{x_0(z)}, y_0^\star(z)\rangle_{Y,Y^\star}|. \tag{6.6.4.4}$$

By (6.6.4.3) and (6.6.4.4) we get

$$\begin{aligned}(1-|z|^2)^{k-\gamma_0}\|R^{\alpha,k}b(z)\|_{\mathcal{L}(\overline{X},Y)} &= \left|\int_{\mathbb{B}_n} \langle b(w)\overline{x_0(z)}, y_0^\star(z)\rangle_{Y,Y^\star} \frac{(1-|z|^2)^{k-\gamma_0}}{(1-\langle z,w\rangle)^{n+1+\alpha}}\right. \\ &= |\langle h_b x_z, y_z^\star\rangle_{\alpha,Y}|,\end{aligned}$$

with

$$x_z(w) = \frac{x_0(z)(1-|z|^2)^{\beta-(n+1+\alpha)/p}}{(1-\langle w,z\rangle)^{\beta}}, \qquad w \in \mathbb{B}_n$$

and

$$y_z^\star(w) = \frac{y_0^\star(z)(1-|z|^2)^{k+(n+1+\alpha)/q-\beta}}{(1-\langle w,z\rangle)^{n+1+\alpha+k-\beta}}, \qquad w \in \mathbb{B}_n,$$

where β is chosen such that

$$(n+1+\alpha)/p < \beta < k + (n+1+\alpha)/q.$$

By Theorem 6.2.10, we have $x_z \in A^p_\alpha(\mathbb{B}_n, X)$, $y_z^\star \in A^{q'}_\alpha(\mathbb{B}_n, Y^\star)$, and

$$\sup_{z\in\mathbb{B}_n} \|x_z\|_{p,\alpha,X} < \infty, \qquad \sup_{z\in\mathbb{B}_n} \|y_z^\star\|_{q',\alpha,Y^\star} < \infty.$$

Let us prove that

$$x_z \longrightarrow 0 \ \text{ weakly in } \ A^p_\alpha(\mathbb{B}_n, X) \ \text{ as } \ |z| \longrightarrow 1^-. \tag{6.6.4.5}$$

Since

$$\sup_{z\in\mathbb{B}_n} \|x_z\|_{p,\alpha,X} < \infty,$$

to prove (6.6.4.5), by Lemma 6.6.9, it suffices to prove that

$$\langle x_z, e_{w,a^\star}\rangle_{\alpha,X} \longrightarrow 0 \quad \text{as} \quad |z| \longrightarrow 1^-,$$

where for each $a^\star \in X^\star$ and $w \in \mathbb{B}_n$, we have

$$e_{w,a^\star}(\zeta) = \frac{1}{(1-\langle \zeta, w\rangle)^{n+1+\alpha}} a^\star, \qquad \zeta \in \mathbb{B}_n.$$

By using the definition of $e_{w,a^\star}$ and the reproducing kernel formula, it follows that

$$\begin{aligned}
\langle x_z, e_{w,a^\star}\rangle_{p,\alpha,X} &= \int_{\mathbb{B}_n} \langle x_z(\zeta), e_{w,a^\star}(\zeta)\rangle_{X,X^\star} \mathrm{d}\nu_\alpha(\zeta) \\
&= \int_{\mathbb{B}_n} \langle x_z(\zeta), \frac{1}{(1-\langle \zeta, w\rangle)^{n+1+\alpha}} a^\star\rangle_{X,X^\star} \mathrm{d}\nu_\alpha(\zeta) \\
&= \Big\langle \int_{\mathbb{B}_n} \frac{x_z(\zeta)}{(1-\langle w, \zeta\rangle)^{n+1+\alpha}} \mathrm{d}\nu_\alpha(\zeta), a^\star \Big\rangle_{X,X^\star} \\
&= \langle x_z(w), a^\star\rangle_{X,X^\star}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|\langle x_z, e_{w,a^\star}\rangle_{p,\alpha,X}| &= |\langle x_z(w), a^\star\rangle_{X,X^\star}| \\
&= \left| \frac{(1-|z|^2)^{\beta-(n+1+\alpha)/p}}{(1-\langle w, z\rangle)^{\beta}} \langle x_0(z), a^\star\rangle_{X,X^\star} \right| \\
&\leq \frac{(1-|z|^2)^{\beta-(n+1+\alpha)/p}}{(1-|w|)^{\beta}} \|a^\star\|_{X^\star} \longrightarrow 0
\end{aligned}$$

as $|z| \longrightarrow 1^-$. By using (6.6.4.5), the compactness of h_b and the fact that

$$\sup_{z\in\mathbb{B}_n} \|y_z^\star\|_{q',\alpha,Y^\star} < \infty,$$

it follows that

$$\lim_{|z|\to 1^-}(1-|z|^2)^{k-\gamma_0}\|R^{\alpha,k}b(z)\|_{\mathcal{L}(\overline{X},Y)} = \lim_{|z|\to 1^-}|\langle h_b x_z, y_z^\star\rangle_{\alpha,Y}| = 0,$$

which completes the proof of the theorem. $\square$

Chapter 7

WEIGHTED INEQUALITIES

7.1 Introduction

Weighted inequalities appeared almost simultaneously with the birth of singular integrals that stimulated their development, in particular the problem of characterisation of positive function ω for which singular integral maps $L^p(\omega d\mu)$ to itself. A famous example of a singular integral is the Bergman projection, whose boundedness problem, solved elsewhere by Békollé and Bonami, is historically linked to the duality problem for Bergman spaces.

For $a > -1$, it is a well-known result of Békollé and Bonami that the Bergman projection T_a, defined by:

$$T_a f(z) := \int_{\mathbb{B}} \frac{f(x)}{(1 - \langle z, x \rangle)^{N+1+a}} d\mu(x)$$

is bounded on $L^p(\omega d\mu_a)$ if and only if the weight ω belongs to the so-called Békollé - Bonami class of weights. The Bergman projection can be extended to all a less than or equal to -1. Therefore a natural question is whether the Békollé - Bonami result can be generalized. In this paper we work with more general operators than the extended Bergman projection, and more generally we characterize weights for which we have the boundedness between two general weighted Lebesgue classes on the unit ball of $\mathbb{C}^N$.

The inner product and the norm in $\mathbb{C}^N$ are $\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_N\overline{w_N}$ and $|z| = \sqrt{\langle z, z\rangle}$. We let $d\mu_q(z) = (1-|z|^2)^q d\mu(z)$ where $q > -1$ and μ be the Lebesgue (volume) measure on the unit ball $\mathbb{B} = \{z \in \mathbb{C}^N : |z| < 1\}$ of $\mathbb{C}^N = \mathbb{R}^{2N}$ normalized with $\mu(\mathbb{B}) = 1$. We set $L^p_q := L^p(d\mu_q)$ the Lebesgue space on $\mathbb{B}$ relative to μ_q with $1 \le p \le +\infty$. Let $H(\mathbb{B})$ to denote the space of holomorphic functions in the unit ball $\mathbb{B}$. For $q > -1$, a function $f \in H(\mathbb{B})$ belongs to the weighted Bergman space A^p_q whenever $f \in L^p(d\mu_q)$. The norm $\|f\|_{A^p_q}$ is simply the L^p_q norm of f.

Besov spaces extend weighted Bergman spaces to all q. To define them, we first take a radial differential operator D^t_s of order t for any $s, t \in \mathbb{R}$ defined on $H(\mathbb{B})$. Let $f \in H(\mathbb{B})$ be given on $\mathbb{B}$ by its convergent homogeneous expansion $f = \sum_{k=0}^{\infty} f_k$ in which f_k is a homogeneous polynomial in $z_1, \ldots, z_N$ of degree k. We define, for $s, t \in \mathbb{R}$

$$D^t_s f := \sum_{k=0}^{\infty} d_k(s,t) f_k = \sum_{k=0}^{\infty} \frac{c_k(s+t)}{c_k(s)} f_k$$

where

$$c_k(a) = \begin{cases} \frac{(N+1+a)_k}{k!} & \text{if} \quad a > -(N+1) \\ \frac{k!}{(1-N-a)_k} & \text{if} \quad a \leq -(N+1) \end{cases}$$

Consider the linear transformation I_s^t defined for $f \in H(\mathbb{B})$ by:

$$I_s^t f(z) = (1-|z|^2)^t D_s^t f(z).$$

We say that a function $f \in H(\mathbb{B})$ belongs to the Besov space B_q^p whenever $I_s^t f \in L_q^p$ for some s, t satisfying:

$$\begin{cases} q + pt > -1 & \text{if} \;\; 1 \leq p < \infty \\ t > 0 & \text{if} \;\; p = \infty. \end{cases}$$

It is well known [109] that the L_q^p-norm, $\|I_s^t f\|_{L_q^p}$, of any one of the functions $I_s^t f$ is an equivalent norm for $\|f\|_{B_q^p}$, the norm of f in B_q^p. When $q > -1$ we have $A_q^p = B_q^p$. The space B_q^2 is a Hilbert space with reproducing kernel K_q (see [109] or [103, Theorem 1.9] or [114]) defined by

$$K_q(z,w) = \begin{cases} \frac{1}{(1-\langle z,w\rangle)^{N+1+q}} & = & \displaystyle\sum_{k=0}^{\infty} \frac{(N+1+q)_k}{k!} \langle z,w\rangle^k, \quad \text{if} \;\; q > \\ {}_2F_1(1,1;1-(N+q);\langle z,w\rangle) & = & \displaystyle\sum_{k=0}^{\infty} \frac{k!}{(1-N-q)_k} \langle z,w\rangle^k, \quad \text{if} \;\; q \leq \end{cases}$$

where ${}_2F_1 \in H(\mathbb{D})$ is the Gauss hypergeometric function and $(u)_v$ is the Pochhammer symbol defined by $(u)_v = \frac{\Gamma(u+v)}{\Gamma(u)}$, where Γ is the Gamma function. Namely, for a number s satisfying $q+1 < p(s+1)$, if t satisfies $q + pt > -1$ then for $f \in B_q^2$ (see [109, Theorem 1.2])

$$(P_s \circ I_s^t) f = \frac{N!}{(1+s+t)_N} f,$$

where

$$P_s f(z) = \int_{\mathbb{B}} K_s(z,w) f(w) (1-|w|^2)^s d\mu(w),$$

is the extended Bergman projection (s may be smaller than or equal to -1).

For $a, b, s, t \in \mathbb{R}$ the operators that we are interested in are defined by (reproducing) Bergman-Besov kernels. For $f \in L^p(d\mu_q)$ we define

$$\begin{aligned}
T^q_{a,b} f(z) &:= T_{a,b} f(z) = \int_{\mathbb{B}} K_a(z,w) f(w)(1-|w|^2)^{b-q} d\mu_q(w), \\
S^q_{a,b} f(z) &:= S_{a,b} f(z) = \int_{\mathbb{B}} |K_a(z,w)||f(w)|(1-|w|^2)^{b-q} d\mu_q(w), \\
P^q_{s,t} f(z) &:= P_{s,t} f(z) = (1-|z|^2)^t \int_{\mathbb{B}} K_{s+t}(z,w) f(w)(1-|w|^2)^{s-q} d\mu_q(w).
\end{aligned}$$

Throughout the paper $b > -1$ and $s > -1$ because we want our operator to be well defined (see for example Lemma 7.5.1 and Lemma 7.5.2). Note that

$$P_{s,t} f(z) = (1-|z|^2)^t T_{s+t,s} f(z). \tag{7.1.0.1}$$

Our main motivation comes from the operators $P_{s,0}$, and $P^+_{s,N+1+s}$, which are the Bergman projection and Berezin transform respectively, where $P^+_{s,N+1+s} f(z) = (1-|z|^2)^t S_{s+N+1+s,s} f(z)$. The operators $P_{s,t}$, $T_{a,b}$ and $S_{a,b}$ are important in the study of function-theoretic operator theory, see for example [115] when $q = -N-1$.

The boundedness of the operators $T^q_{a,b}$ was already studied by Kaptanoglu and Ureyen [110] in the cases where the operators $T^q_{a,b}$ act from L^p_q to L^P_Q, with $q \in \mathbb{R}, 1 \leq p, P \leq \infty, Q > -1$.

Theorem 7.1.1. *[110, Theorem 1.2] Let $a, b, q, Q \in \mathbb{R}$, $1 \leq p \leq P \leq \infty$, and assume $Q > -1$ when $P < \infty$. Then the following three conditions are equivalent*

1. *$T_{a,b} : L^p_q \to L^P_Q$;*

2. *$S_{a,b} : L^p_q \to L^P_Q$;*

3. (a) $\frac{1+q}{p} < 1+b$ *and* $a \leq b + \frac{1+N+Q}{P} - \frac{1+N+q}{p}$ *for* $1 < p \leq P < \infty$;

(b) $\frac{1+q}{p} \leq 1+b$ *and* $a \leq b + \frac{1+N+Q}{P} - \frac{1+N+q}{p}$ *for* $1 = p \leq P \leq \infty$, *but at least one inequality must be strict;*

(c) $\frac{1+q}{p} < 1+b$ *and* $a < b + \frac{1+N+Q}{P} - \frac{1+N+q}{p}$ *for* $1 < p \leq P = \infty$.

This result is useful for our work, especially for the case $p = P$ and $q = Q$, to investigate the case where these operators $P_{s,t}$ and $T_{a,b}$ are bounded from L^1_q to $L^{1,\infty}_q$. Our main result in this direction is the following.

Theorem 7.1.2. *In the case $q = s$, $s + 2t > -1$ and $s + t > -1$ with $s > -1$ the operators $P_{s,t}$ are bounded from L^1_q to $L^{1,\infty}_q$ and not from L^1_q to L^1_q.*

In this paper we also investigate the more general cases with weights ω for the boundedness of $T_{a,b}$ and $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. In the special case of the Bergman projection $T_{a,0}$, Békollé [104] obtained the characterisation of the weights ω in terms of the Békollé -Bonami condition.

Let d be the pseudo-distance in $\mathbb{B}$ defined by $d(z,w) = ||z|-|w|| + \left|1 - \frac{\langle z,w\rangle}{|z||w|}\right|$.

Definition 7.1.3 (Békollé -Bonami class)**.** Let ω be a locally integrable non negative function on $\mathbb{B}$ (a weight). We say that $\omega d\mu_a$ belongs to (B_p), $1 < p < \infty$, if there is a constant $C_p(\omega)$ such that for every ball B (with respect to the pseudo-distance d) of $\mathbb{B}$ that intersects the closure of $\mathbb{B}$, we have

$$\frac{\omega d\mu_a(B)}{\mu_a(B)} \left(\frac{1}{\mu_a(B)} \int_{\mathbb{B}} \omega^{\frac{-1}{p-1}} d\mu_a \right)^{p-1} \leq C_p(\omega).$$

For $a > -1$, let

$$T_a f(z) := T_{a,0} f(z) := \int_{\mathbb{B}} \frac{f(x)}{(1 - \langle z, x \rangle)^{N+1+a}} d\mu_a(x)$$

be the Bergman projection. Békollé showed in [104] that

Theorem 7.1.4. *Let ω be a locally integrable non negative function on $\mathbb{B}$. The operator T_a, $a > -1$, is well defined and continuous on $L^p(\omega d\mu_a)$, $1 < p < \infty$, if and only if $\omega d\mu_a \in (B_p)$.*

The results we obtain depend upon the values of a, $s+t$, q and Q. In the case $a < -(N+1)$, $s+t < -(N+1)$ we have the two following main results:

Theorem 7.1.5. *In the case $a < -(N+1)$, $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ if and only if*

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_Q(z) \right) \left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z) \right)^{p-1} < \infty.$$

Moreover

$$\|T_{a,b}\|^p \simeq \left(\int_{\mathbb{B}} \omega(z) d\mu_Q(z) \right) \left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z) \right)^{p-1}.$$

Theorem 7.1.6. *In the case $s+t < -(N+1)$, there are no weights ω such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \leq q$.*

Theorem 7.1.7. *In the case $s+t < -(N+1)$, if $Q > q$, then $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ if and only if*

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z) \right) \left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty.$$

Moreover

$$\|P_{s,t}\|^p \simeq \left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z) \right) \left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1}.$$

In order to give our necessary condition for the boundedness of $T_{a,b}$ when $a > -(1+N)$ we introduce a Békollé -Bonami type class of weights denoted by $(B_p^{a,b,q,Q})$.

Definition 7.1.8. For $Q \leq q$ and $a > -1$, we say that $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{\mu_b(B)}{\mu_a^2(B)} \int_B \omega(z) d\mu_Q(z)\right) \left(\frac{\mu_b(B)}{\mu_a^2(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z)\right)^{p-1} < \infty$$

where the supremum is taken over the pseudoballs B.

For $Q \leq q$ and $a > -N-1$, we say that $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{\mu_b(B)}{R_B^{2(N+1+a)}} \int_B \omega(z) d\mu_Q(z)\right) \left(\frac{\mu_b(B)}{R_B^{2(N+1+a)}} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z)\right)$$

where the supremum is taken over the pseudoballs B with radius R_B.

For $Q > q$ and $a > -1$, we say that $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)} \int_B \omega(z) d\mu_Q(z)\right) \left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z)\right.$$

where the supremum is taken over the pseudoballs B.

For $Q > q$ and $a > -N-1$, we say that $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R_B^{2(N+1+a)}} \int_B \omega(z) d\mu_Q(z)\right) \left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R_B^{2(N+1+a)}} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z)\right.$$

where the supremum is taken over the pseudoballs B with radius R_B.

The necessary condition for the boundedness of $T_{a,b}$ when $a > -(1+N)$ is the following.

Theorem 7.1.9. *Suppose that $-(1+N) < a$ and $b > -1$. If $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, then we have $\omega \in (B_p^{a,b,q,Q})$.*

For $P_{s,t}$ we demonstrate

Theorem 7.1.10. *In the case both $-(N+1) < s+t < -1$ and $s+t+\frac{Q-q}{p} \le -1$ hold, and in the case both $s+t > -1$ and $Q < q$ hold, there are no weights ω such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$.*

For the remaining cases, we introduce $(K_p^{s,t,q,Q})$ another Békollé -Bonami type class of weights in order to give our necessary condition for the boundedness of $P_{s,t}$ when $s+t > -(1+N)$.

Definition 7.1.11. For $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$, we say that $\omega \in (K_p^{s,t,q,Q})$ $(s > -1)$ if

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{R_B^{\frac{Q-q}{p}}}{R_B^{N+1+s+t}}\int_B \omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{R_B^{\frac{Q-q}{p}}}{R_B^{N+1+s+t}}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(

The necessary condition for the boundedness of $P_{s,t}$ when $s+t > -(1+N)$ is the following.

Theorem 7.1.12. *In the case both $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$ hold, and in the case both $s+t > -1$ and $Q \geq q$ hold, if $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, then $\omega \in (K_p^{s,t,q,Q})$.*

We introduce a maximal and a fractional maximal operator that will be used to establish a good lambda inequality in order to give sufficient conditions for the boundedness of $P_{s,t}$.

If $a > -1$ we set:

$$m_{a,b}f(z) = \sup_{\zeta \in \mathbb{B}, R > 1-|\zeta| : z \in B(\zeta,R)} \frac{1}{\mu_a(B(\zeta,R))} \int_{B(\zeta,R)} |f(w)| d\mu_b(w),$$

more generally if $a > -1-N$ we set:

$$m'_{a,b}f(z) = \sup_{\zeta \in \mathbb{B}, R > 1-|\zeta| : z \in B(\zeta,R)} \frac{1}{R^{N+1+a}} \int_{B(\zeta,R)} |f(w)| d\mu_b(w).$$

Before giving our good lambda inequality, we introduce here $(D_p^{s,t,q,Q})$ a Békollé-Bonami type class of weights.

Definition 7.1.13. For $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$, we say that $\omega \in (D_p^{s,t,q,Q})$ $(s > -1)$ if

$$\sup_{B : B \cap \partial\mathbb{B} \neq \varnothing} \left(\frac{1}{R_B^{N+1+s+t+\frac{Q-q}{p}}} \int_B \omega(z) d\mu_{Q+pt}(z) \right) \left(\frac{1}{R_B^{N+1+s+t+\frac{Q-q}{p}}} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu$$

where the supremum is taken over the pseudoballs B with radius R_B.

For $Q \geq q$ and $s+t > -1$, we say that $\omega \in (D_p^{s,t,q,S})$ $(s > -1)$ if

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)}\int_B \omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(}$$

where the supremum is taken over the pseudoballs B with radius R_B. In each case we denote by $D_p^{s,t,q,Q}(\omega)$ the expression in the left hand side.

Remark 7.1.14. Constants and standard weights ($\omega(z) = (1-|z|^2)^\eta$) are in $(D_p^{s,t,q,Q})$. We also have $(D_p^{s,t,q,q}) \subseteq (D_p^{s,t,q,Q}) \subseteq (K_p^{s,t,q,Q})$. For $Q = q$ we have $(K_p^{s,t,q,Q}) = (D_p^{s,t,q,Q})$.

Here is our good lambda inequality.

Theorem 7.1.15. *Suppose that* $1 < p < \infty$. *Let* $\omega \in (D_p^{s,t,q,Q})$ *where both* $s+t+\frac{Q-q}{p} > -1$ *and* $-1 > s+t > -N-1$ *hold, or both* $s+t > -1$ *and* $Q \geq q$ *hold. There are two positive constants* C *and* β *such that for all* γ *sufficiently small,* $\lambda > 0$ *and for all positive locally integrable functions* f *if* $-N-1 < s+t$ *and* $s+t+\frac{Q-q}{p} > -1$, *then*

$$\omega d\mu_{Q+pt}(\{z\in\mathbb{B} : S_{s+t,s}f(z) > 2\lambda, m'_{s+t,s}f(z)\leq\gamma\lambda\}) \leq$$
$$CD_p^{s,t,q,Q}(\omega)\gamma^\beta\omega d\mu_{Q+pt}(\{z\in\mathbb{B} : S_{s+t,s}f(z) > \lambda\}). \quad (7.1.0.2)$$

To show that $(D_p^{s,t,q,Q})$ is sufficient for the boundedness of $P_{s,t}$ when $s+t > -1$, we introduce the following maximal and fractional maximal operator. If $s+t > -1$ we set:

$$O_{s,t}f(z) = (1-|z|^2)^t m_{s+t,s}f(z);$$

more generally if $s+t > -1-N$ we set:

$$O'_{s,t}f(z) = (1-|z|^2)^t m'_{s+t,s}f(z).$$

The following theorem shows together with the good lambda inequality that $(D_p^{s,t,q,Q})$ is sufficient for the boundedness of $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ when $-N-1 < s+t$ and $s+t+\frac{Q-q}{p} > -1$:

Theorem 7.1.16. *For $-N-1 < s+t$ and $s+t+\frac{Q-q}{p} > -1$, if $\omega d\mu_q \in (D_p^{s,t,q,Q})$, there is a constant $C_{s,t,p,q,Q} > 0$ such that $\forall f \in L^p(\omega d\mu_q)$,*

$$\int_{\mathbb{B}} (O_{s,t}f(z))^p \omega(z) d\mu_q(z) \leq C_{s,t,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z).$$

Then for the case $s+t > -1$ and $q = Q$ we have:

Corollary 7.1.17. *Let ω be a weight on $\mathbb{B}$. Then for $s+t > -1, s > -1$ the following assertions are equivalent:*

1. *$P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_q)$;*

2. *$T_{s+t,s}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_{q+pt})$;*

3. *$S_{s+t,s}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_{q+pt})$;*

4. *$\omega \in (K_p^{s,t,q,Q})$.*

Rahm, the third and the fourth author in [111] settled the particular case of the operators $P_{s,t}$ for $s+t > -1, s > -1, Q = q = s$. To this aim, they used dyadic methods that have been initiated by Aleman, Pott and Reguera in the unit disk [102].

The outline of the paper is as follows. In Section 2 we briefly give requisite background information. We prove Theorem 7.1.2 in Section 3. From Section 4 we look at weighted estimates; there we show Theorem 7.1.5, Theorem 7.1.6 and Theorem 7.1.7. The proof of Theorem 7.1.9, Theorem 7.1.10, Theorem 7.1.12, Theorem 7.1.16 are in Section 5. The proof of Theorem 7.1.15 is in Section 6. Corollary 7.1.17 appears in Section 7.

7.2 Fundamental principles

7.2.1 Complex Analysis techniques

Throughout this paper d is the pseudo-distance defined by

$$d(z,w) = ||z| - |w|| + \left|1 - \frac{\langle z, w\rangle}{|z||w|}\right|.$$

Throughout this paper K will be a constant such that

$$d(x,y) \leqslant K(d(x,z) + d(z,y))$$

for all x, y and z in $\mathbb{B}$. One can find the following two results in [104].

Lemma 7.2.1. *For each $z \in \mathbb{B}$ and r_0, $0 < r_0 < 1$, if we set $z^0 = (r_0, 0, \cdots, 0)$, then we have:*

1. $|1 - z_1 r_0| \geq \frac{1}{4} d(z, z^0)$;
2. $|z_1 - r_0| \leq d(z, z^0)$;
3. $|z - z^0| \leq d(z, z^0)$;
4. $\displaystyle\sum_{k=2}^{N} |z_k|^2 \leq 2d(z, z^0)$.

Proposition 7.2.2. *There is a constant $C_1 > 0$ such that for all $z, w, w_0 \in \mathbb{B}$ such that $d(z, w_0) > C_1 d(w, w_0)$ we have*

$$|\langle z, w_0\rangle - \langle z, w\rangle| \leq \frac{1}{2}|1 - \langle z, w_0\rangle|.$$

Then

$$|1 - \langle z, w\rangle| \geq \frac{1}{2}|1 - \langle z, w_0\rangle|.$$

The following result will be heavily used throughout the paper.

Lemma 7.2.3. *For each* $w \in \mathbb{B}$, $0 < |w| = r < 1$ *and* $0 < R < 2$*:*

$$\mu_q(B(w,R)) \simeq R^{N+1}[max(R, 1-r)]^q \ if \ q > -1.$$

Then for $q > -1$, $(\mathbb{B}, d, \mu_q)$ *is an homogeneous space in the sense of [106].*

However, if $B(w,R)$ *is away from the boundary* ($R < \frac{(1-|w|)}{2}$)*, the equivalence remains true if* $q < -1$*, and*

$$\mu_{-1}(B(w,R)) \simeq R^N.$$

Proof. We are going to do the case $q < -1$ and $q = -1$, one can find the other case in [104].

First case: Assume $q < -1$.

We show that for all $R \in (0, \frac{1-|w|}{2})$ We have

$$\int_{B(w,R)} (1-|z|)^q d\mu(z) \simeq R^{N+1}(1-|w|)^q.$$

We have

$$\{z \in \mathbb{B} : ||z| - |w|| \leq \frac{R}{2} \text{ and } |1 - \langle z', w' \rangle| \leq \frac{R}{2}\} \subset B(w,R) \subset \{z \in \mathbb{B} : ||z| - |w|| \leq R$$

where $z' = \frac{z}{|z|}, w' = \frac{w}{|w|}$. We first of all recall that $\int_{\{|1-\langle z', w'\rangle| \leq R, z' \in \partial\mathbb{B}\}} \mathrm{d}\sigma(z') \simeq R^N$ (see [115]).

$\mu_q(B(w,R)) = \int_{B(w,R)} (1-|z|)^q d\mu(z) \leq \int_{\{z \in \mathbb{B} : ||z|-|w|| \leq R \text{ and } |1-\langle z', w'\rangle| \leq R\}} (1-|z|)^q d\mu(z).$

We set

$$Y = \int_{\{z\in\mathbb{B}:||z|-|w||\leq R \text{ and } |1-\langle z',w'\rangle|\leq R\}} (1-|z|)^q d\mu(z).$$

Then,

$$\begin{aligned}
Y &\lesssim \int\limits_{|w|-R<\rho<|w|+R} (1-\rho^2)^q \rho^{2N-1}\mathrm{d}\rho \int\limits_{\{|1-\langle z',w'\rangle|\leq R, z'\in\partial\mathbb{B}\}} \mathrm{d}\sigma(z') \\
&\lesssim R^N \int\limits_{|w|-R<\rho<|w|+R} (1-\rho)^q \mathrm{d}\rho \\
&= -\frac{R^N}{q+1}\{(1-|w|-R)^{q+1} - (1-|w|+R)^{q+1}\} \\
&= -\frac{R^N}{q+1}(q+1)(-2R)(1-|w|-\theta R)^q \\
&\simeq R^{N+1}(1-|w|)^q,
\end{aligned}$$

where the last equivalence is due to the fact that $\theta R \leq R < \frac{1-|w|}{2}$.

In the same way we get $\mu_q(B(w,R)) \gtrsim R^{N+1}(1-|w|)^q$, using this time the fact that

$$\left\{z\in\mathbb{B} : ||z|-|w|| \leq \frac{R}{2} \text{ and } |1-\langle z',w'\rangle| \leq \frac{R}{2}\right\} \subset B(w,R).$$

Second case: Assume $q=-1$.

For this case, notice that

$$\int\limits_{|w|-R<\rho<|w|+R} (1-\rho)^{-1}\mathrm{d}\rho \simeq \left[\ln(\frac{1}{1-\rho})\right]_{|w|-R}^{|w|+R} = \ln(\frac{1-|w|+R}{1-|w|-R}) \leq \ln 3.$$

Then $\mu_{-1}(B(w,R)) \simeq R^N$. $\square$

The following result can be found in [110].

Lemma 7.2.4. *1. For $q < -(N+1)$, each $|K_q(z,w)|$ is bounded above as z, w vary in $\mathbb{B}$.*

2. For each $q \in \mathbb{R}$,

(a) $|K_q(z,w)|$ is bounded below by a positive constant as z, w vary in $\mathbb{B}$. In particular, $K_q(z,w)$ is zero free in $\mathbb{B} \times \mathbb{B}$.

(b) there is a $\rho_0 < 1$ such that for $|z| \leq \rho_0$ and all $w \in \mathbb{B}$, we have $\Re K_q(z,w) \geq \frac{1}{2}$.

Proof. We recall that

$$K_q(z,w) = \begin{cases} \frac{1}{(1-\langle z,w\rangle)^{N+1+q}} & = \displaystyle\sum_{k=0}^{\infty} \frac{(N+1+q)_k}{k!} \langle z,w\rangle^k, \quad \text{if } q > \\ {}_2F_1(1,1;1-(N+q);\langle z,w\rangle) & = \displaystyle\sum_{k=0}^{\infty} \frac{k!}{(1-N-q)_k} \langle z,w\rangle^k, \quad \text{if } q \leq \end{cases}$$

so $K_q(z,w) = \sum_{k=0}^{\infty} c_k(q)\langle z,w\rangle^k = \sum_{k=0}^{\infty} c_k(q) v^k = k_q(v)$ where $v = \langle z,w\rangle$. By Stirling's formula $c_k(q) \sim k^{N+q} \quad (k \to \infty)$, so that when $q < -(N+1)$, the power series of $k_q(v)$ converges uniformly for $v \in \overline{\mathbb{D}}$. This shows boundedness. When $q < -(N+1)$, see that k_q is not zero on a set containing $\overline{\mathbb{D}} - \{1\}$. The reason for this is that the first term 1 (for $k = 0$) of the hypergeometric function k_q is positive. But also $k_q(1) \neq 0$. Thus $|k_q|$ for $q < -(1+N)$ is bounded below on $\overline{\mathbb{D}}$.

If $q = -(1+N)$, then $k_{-(1+N)}(v) = v \log(1-v)^{-1}$. On $\overline{\mathbb{D}} - \{1\}$, $k_{-(1+N)}$ is not zero and $|k_{-(1+N)}(v)|$ blows up as $v \to 1$ within $\overline{\mathbb{D}}$. So $|k_{-(1+N)}|$ is bounded below on $\overline{\mathbb{D}}$.

The claim about $|k_q|$ for $q > -(1+N)$ is obvious and the lower bound can be taken as $2^{-(1+N+q)}$. Then (1) follows.

Finally, $|K_q(z,w)| \le 1 + C\sum_{k=1}^{\infty} k^{N+q}|\langle z,w\rangle|^k$ for some constant C and

$$C\sum_{k=1}^{\infty} k^{N+q}|\langle z,w\rangle|^k \le C\sum_{k=1}^{\infty} k^{N+q}|z|^k|w|^k \le C|z|\sum_{k=1}^{\infty} k^{N+q}|z|^{k-1}$$

for all $z, w \in \mathbb{B}$. The last series converges, say, for $|z| = \frac{1}{2}$; call its sum W and set $\rho_0 = \min\{\frac{1}{2}, \frac{1}{2CW}\}$. If $|z| \le \rho_0$, then

$$|C\sum_{k=1}^{\infty} k^{N+q}\langle z,w\rangle^k| \le CW|z| \le \frac{1}{2} \quad (z \in \mathbb{B}).$$

This is, $|K_q(z,w) - 1| \le \frac{1}{2}$ for $|z| \le \rho_0$ and all $z \in \mathbb{B}$. This implies the desired result (2). □

One can find this in [115].

Proposition 7.2.5. *Let*

$$I = \int_{\mathbb{B}} \frac{(1-|w|^2)^d}{|1-\langle z,w\rangle|^{1+N+c}} d\mu(w),$$

for $d > -1$ and $c \in \mathbb{R}$. We have:

(i) $I \sim 1$ if $c < d$;

(ii) $I \sim \frac{1}{|z|^2}\log\frac{1}{1-|z|^2}$ if $c = d$;

(iii) $I \sim (1-|z|^2)^{-(c-d)}$ if $c > d$.

Theorem 7.2.6. *Equipped with the following equivalent scalar product*

$${}_q\langle f,g\rangle_s^t = \int_{\mathbb{B}} I_s^t f(z)\overline{I_s^t g(z)}\,d\mu_q(z), \quad q + 2t > -1,$$

B_q^2 is a Hilbert space with reproducing kernel given by:

$$K_q(z,w) = \begin{cases} \frac{1}{(1-\langle z,w\rangle)^{N+1+q}} = \sum_{k=0}^{\infty} \frac{(N+1+q)_k}{k!}\langle z,w\rangle^k, & \text{if } q > \\ {}_2F_1(1,1;1-(N+q);\langle z,w\rangle) = \sum_{k=0}^{\infty} \frac{k!}{(1-N-q)_k}\langle z,w\rangle^k, & \text{if } q \le \end{cases}$$

7.2.2 Harmonic Analysis techniques

The following result can be found in [106] and will be helpful in the proof of Theorem 7.3.1.

Theorem 7.2.7 (Coifman and Weiss, [106]). *Let (X, d, μ) be an homogeneous space and let $K(x, y)$ be a function such that $K(x, .) : y \to K(x, y) \in L^2(X)$. If the operator T defined by*

$$Tf(x) = \int_X K(x, y)f(y)d\mu(y),$$

satisfies the following two conditions:

1. *there is a constant C_1 such that $\|Tf\|_2 \leq C_1\|f\|_2$;*

2. *there are two constants C_2 and C_3 such that for all y, y_0 we have:*

$$\int_{d(x,y_0)>C_2d(y,y_0)} |K(x, y)-K(x, y_0)|d\mu(x) < C_3, \quad \text{(Hörmander Condition)}$$

then for all p, $1 \leq p \leq 2$, there is a constant A_p depending only on $C_i, i = 1, 2, 3$, such that for all $f \in L^2 \bigcap L^p$ we have: $\|Tf\|_p \leq A_p\|f\|_p$ if $p > 1$ and $\forall\lambda > 0 :$

$$\mu(\{x \in X : |Tf(x)| > \lambda\}) \leq A_1 \frac{\|f\|_1}{\lambda}.$$

One can find the following result in [108].

Theorem 7.2.8 (Marcinkiewicz Interpolation Theorem). *Let p_0, p_1 be such that $1 \leq p_0 < p_1 \leq \infty$. Let T be a sublinear operator defined from $L^{p_0} + L^{p_1}$ to the space of measurable functions. Assume that T is simultaneously of weak type (p_0, p_0) with operator norm A_{p_0,p_0} and of weak type (p_1, p_1) with operator norm A_{p_1,p_1}. Then for every $0 < t < 1$, T is of (strong) type (p_t, p_t) where*

$$\frac{1}{p_t} = \frac{t}{p_0} + \frac{1-t}{p_1}.$$

Moreover, if $p_1 < \infty$*, then* $\|Tf\|_{p_t} \le A_{p_t,p_t}\|f\|_{p_t}$ *with*

$$A_{p_t,p_t} = 2\left[p_t\left(\frac{A^{p_0}_{p_0,p_0}}{p_t - p_0} - \frac{A^{p_1}_{p_1,p_1}}{p_1 - p_t}\right)\right]^{\frac{1}{p_t}}.$$

If $p_1 = \infty$*, we can take*

$$A_{p_t,p_t} = 2\left[p_t\frac{A^{p_0}_{p_0,p_0}}{p_t - p_0}\right]^{\frac{1}{p_t}}.$$

The fractional maximal function is defined as follows:

$$M_\gamma f(z) = \sup_{B:z\in B}\frac{1}{\nu^{1-\gamma}(B)}\int_B |f(w)|d\nu(w), \quad \gamma \in [0,1).$$

When $\gamma = 0$ it is the Hardy-Littlewood maximal operator. The following result will be used in Section 7.5 in the study of our maximal and fractional maximal function. One can find their proof in [107] or in [112].

Theorem 7.2.9. *Let* X *an homogeneous space,* $0 \le \gamma < 1$*,* $1 < p \le r < \infty$ *and a pair of weights* (u, v)*, then the following are equivalent:*

(i) there exists a constant $C_1 > 0$ *such that*

$$\left(\int_X [M_\gamma f(x)]^r v(x)d\nu(x)\right)^{\frac{1}{r}} \le C_1\left(\int_X |f(x)|^p u(x)d\nu(x)\right)^{\frac{1}{p}}$$

for any $f \in L^p(X, ud\nu)$*;*

(ii) there exists a constant $C_2 > 0$ *such that*

$$\left(\int_B [M_\gamma(\chi_B u^{1-p'})(x)]^r v(x)d\nu(x)\right)^{\frac{1}{r}} \le C_2\left(\int_B u^{1-p'}(x)d\nu(x)\right)^{\frac{1}{p}}$$

for any ball $B \subset X$*.*

We will also make use of the following class, in Section 7.5, in the study of our maximal and fractional maximal function and to establish the good lambda inequality.

Definition 7.2.10. A measure $\omega d\mu_\alpha$ is in the (A_p, α) $(1 < p < \infty)$ class if there is a constant $C_p(\omega)$ such that for all pseudo-ball $B := B(\zeta, R)$ we have:

$$\left(\frac{1}{\mu_\alpha(B)}\int_B \omega(z)d\mu_\alpha(z)\right)\left(\frac{1}{\mu_\alpha(B)}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_\alpha(z)\right)^{p-1} \leq C_p(\omega).$$

Definition 7.2.11. A measure $\omega d\mu_\alpha$is a Muckenhoupt weight or is in the (A_∞, α) class if for all δ such that $0 < \delta < 1$, there is β, $0 < \beta < 1$, such that for all pseudo balls B of $\mathbb{B}$ and for all measurable subset E of $\mathbb{B}$ we have:

$$\mu_\alpha(E) \geq \delta\mu_\alpha(B) \Rightarrow \omega d\mu_\alpha(E) \geq \beta\omega d\mu_\alpha(B).$$

We give now two properties of Muckenhoupt weight that we will need later (see [108]).

Lemma 7.2.12. *If $\sigma \in (A_{\infty,\alpha})$ then there are two positive constants, A and β_0 such that for all ball B and a measurable subset E of B we have:*

$$\sigma d\mu_\alpha(E) \leq A\left(\frac{d\mu_\alpha(E)}{d\mu_\alpha(B)}\right)^{\beta_0}\sigma d\mu_\alpha(B).$$

Theorem 7.2.13. *The Hardy-Littlewood maximal operator is bounded on $L^p(\omega d\mu_\alpha$ $p < \infty$, if and only if $\omega d\mu_\alpha \in (A_p, \alpha)$.*

The following known lemma will be use in Section 7.6.

Lemma 7.2.14. *Let $(X, \mathcal{A}, \mu)$ a measure space. Let f and g two positive measurable functions such that for all $t > 0$*

$$\mu(\{x \in X : f(x) > t, g(x) \leq ct\}) \leq a\mu(\{x \in X : f(x) > bt\}),$$

where a, b and c are positive constants such that $a < b^p$ $(1 < p < \infty)$. Then

$$\|f\|_p^p \leq \frac{c^{-p}}{1 - ab^{-p}}\|g\|_p^p.$$

We will use the following lemma in Section 7.5 to show Lemma 7.5.1 and Lemma 7.5.2. One can find it in [104, 113].

Lemma 7.2.15. *Let $a > -1-N$, there are two constants $C_1, C_2 (C_1 > 0)$ such that $\forall z, w, w_0 \in \mathbb{B}$ such that $|1-\langle z, w_0\rangle| > C_1 d(w, w_0)$, then:*

$$\left|\frac{1}{(1-\langle z, w\rangle)^{N+1+a}} - \frac{1}{(1-\langle z, w_0\rangle)^{N+1+a}}\right| \leq C_2 \frac{d(w, w_0)}{|1-\langle z, w_0\rangle|^{N+a+2}}.$$

7.3 Weak Type L^1 Inequality for $P_{s,t}$ and $T_{a,b}$

In this section, we will prove Theorem 7.1.2 that we first recall here.

Theorem 7.3.1. *In the case $q = s$, $s+2t > -1$ and $s+t > -1$ with $s > -1$ the operators $P_{s,t}$ are bounded from L^1_q to $L^{1,\infty}_q$ and not from L^1_q to L^1_q.*

Proof. The kernel of $P_{s,t}$ is $H_{s,t}(z, w) = \frac{(1-|z|^2)^t}{(1-\langle z,w\rangle)^{N+1+s+t}}$. We are going to proceed in three steps.

Step 1: Show that $H_{s,t}(z, .) \in L^2_q$, $\forall z \in \mathbb{B}$.

Indeed, we have

$$\begin{aligned}\int_{\mathbb{B}} |H_{s,t}(z, w)|^2 d\mu_q(w) &= \int_{\mathbb{B}} \frac{(1-|z|^2)^{2t}}{|1-\langle z, w\rangle|^{2(N+1+s+t)}} d\mu_q(w) \\ &\leq \frac{(1-|z|^2)^{2t}}{(1-|z|)^{2(N+1+s+t)}} \int_{\mathbb{B}} (1-|w|^2)^q d\mu(w)\end{aligned}$$

where in the second inequality, the member of the right hand side is finite because $q = s > -1$.

Step 2: Show that $P_{s,t}$ is bounded from L^2_q to L^2_q.

We have to show the boundedness of $T_{s+t,s}$ from L^2_q to L^2_{q+2t}. By Kaptanoglu and Ureyen, for $a = s + t$, $b = s$, $p = P = 2$ and $Q = q + 2t$ (this is the reason $q + 2t > -1$ is needed), this holds.

Step 3: Show that there are two constants C_1 and C_2 such that $\forall w, w_0 \in \mathbb{B}$ we have:

$$\int_{d(z,w_0)>C_1 d(w,w_0)} |H_{s,t}(z,w) - H_{s,t}(z,w_0)| d\mu_q(z) < C_2.$$

This was already done in [104] (see the proof of [104, Proposition 1] choose $a = q + t + 1$).

Because of *Step 1*, *Step 2* and *Step 3* we have by using Theorem 7.2.7 that the operators $P_{s,t}$ are bounded from L^1_q to $L^{1,\infty}_q$. Observe that $P_{s,t}$ is bounded from L^1_q to L^1_q if and only if $T_{s+t,s}$ is bounded from L^1_q to L^1_{q+t}; and by Theorem 7.1.1, $T_{s+t,s}$ is not bounded from L^1_q to L^1_{q+t} because $q = s$. $\square$

Remark 7.3.2. In the case $a > -(N+1)$, the operators $T^q_{a,b}$ are bounded from L^1_q to $L^{1,\infty}_q$ if we have the following two conditions:

i) $a \leq b$

ii) $-1 < q \leq b$.

The case $a = b = q > -1$ is due to Békollé in [104] and the remaining cases is by Theorem 7.1.1.

Remark 7.3.3. In the special case $b = q$, $T^q_{a,b}$ is self adjoint and bounded from

L^p_q to itself for $1 < p < \infty$. Indeed, let $f \in L^p_q$ and $g \in L^{p'}_q$. Then

$$\begin{aligned}
\langle T^q_{a,b} f, g\rangle_{L^2_q} &= \int_{\mathbb{B}} \int_{\mathbb{B}} K_a(z,w) f(w)(1-|w|^2)^{b-q} d\mu_q(w) \overline{g(z)} (1-|z|^2)^q d\mu(z) \\
&= \int_{\mathbb{B}} f(w) \overline{(1-|w|^2)^{b-q} \int_{\mathbb{B}} K_a(w,z) g(z)(1-|z|^2)^q d\mu(z)} (1-|w|^2)^q d\mu(w) \\
&= \int_{\mathbb{B}} f(w) \overline{T^*_{a,b} g(w) d\mu_q(w)} \\
&= \langle f, (T^q_{a,b})^* g\rangle_{L^2_q},
\end{aligned}$$

where

$$(T^q_{a,b})^* g(w) = (1-|w|^2)^{b-q} \int_{\mathbb{B}} K_a(w,z) g(z)(1-|z|^2)^q d\mu(z).$$

Observe that when $b = q$, $(T^q_{a,b})^* = T^q_{a,b}$ and since $T^q_{a,b}$ is bounded from L^p_q to L^p_q when $1 < p < 2$, then $T^q_{a,b} = \left(T^q_{a,b}\right)^*$ is bounded from $L^{p'}_q$ to $L^{p'}_q$ with $2 < p' < \infty$.

Remark 7.3.4. By Remark 7.3.2 we have that $T^q_{a,b}$ is of weak type $(1,1)$ and let $A_{1,1}$ be the operator norm. By Theorem 7.1.1 we have that $T^q_{a,b}$ is of (weak) type $(2,2)$ and let $A_{2,2}$ be the operator norm. Applying Theorem 7.2.8 leads us, with a better estimation of the operator norm

$$A_{p,p} = 2\left[p\left(\frac{A_{1,1}}{p-1} - \frac{A^2_{2,2}}{2-p}\right)\right]^{\frac{1}{p}},$$

to a new way to have the boundedness of $T^q_{a,b}$ from L^p_q to L^p_q when $1 < p < 2$. In the special case $b = q$ we have the boundedness from L^p_q to L^p_q when $1 < p < \infty$ because in this case $T^q_{a,b}$ is self adjoint.

7.4 Weighted estimates

In this section we will give a proof of our criterion for the weights that provide boundedness of $T_{a,b}$ when $a < -n-1$ (respectively $P_{s,t}$ when $s+t < -N-1$).

We start first with some general necessaries conditions.

7.4.1 Preliminary Necessary Conditions

Lemma 7.4.1. *For $q, Q \in \mathbb{R}$, if $T_{a,b}$ $(a, b \in \mathbb{R})$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, then ω must be in $L^1(d\mu_Q)$.*

Proof. Let $f(w) = (1-|w|^2)^{-b}\chi_{B(0,R)}(w)$ where $B(0,R)$ is the Euclidian ball. Then

$$\begin{aligned}
T_{a,b}f(w) &= \int_{\mathbb{B}} K_a(w,z)(1-|z|^2)^{-b}\chi_{B(0,R)}(z)(1-|z|^2)^b d\mu(z) \\
&= \int_{B(0,R)} K_a(w,z)d\mu(z) \\
&= \overline{\int_{B(0,R)} K_a(z,w)d\mu(z)} \\
&= \overline{K_a(0,w)}\mu(B(0,R)) \\
&= \mu(B(0,R)).
\end{aligned}$$

Since $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$,

$$\int_{\mathbb{B}} |T_{a,b}f(z)|^p \omega(z) d\mu_Q(z) < \infty,$$

so that,

$$\mu^p(B(0,R)) \int_{\mathbb{B}} \omega(z) d\mu_Q(z) < \infty,$$

then

$$\int_{\mathbb{B}} \omega(z) d\mu_Q(z) < \infty,$$

and $\omega \in L^1(d\mu_Q)$. $\square$

Lemma 7.4.2. *For $q, Q \in \mathbb{R}$, if $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, then ω must be in $L^1(d\mu_{Q+pt})$.*

Proof. Let $f(w) = (1-|w|^2)^{-s}\chi_{B(0,R)}(w)$ where $B(0,R)$ is the Euclidian ball. Then

$$\begin{aligned}
P_{s,t}f(w) &= (1-|w|^2)^t \int_{\mathbb{B}} K_{s+t}(w,z)(1-|z|^2)^{-s}\chi_{B(0,R)}(z)(1-|z|^2)^s d\mu(z) \\
&= (1-|w|^2)^t \int_{B(0,R)} K_{s+t}(w,z)d\mu(z) \\
&= (1-|w|^2)^t \overline{\int_{B(0,R)} K_{s+t}(z,w)d\mu(z)} \\
&= (1-|w|^2)^t \overline{K_{s+t}(0,w)}\mu(B(0,R)) \\
&= (1-|w|^2)^t \mu(B(0,R)).
\end{aligned}$$

Since $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$,

$$\int_{\mathbb{B}} |P_{s,t}f(z)|^p \omega(z)d\mu_Q(z) < \infty,$$

so that,

$$\mu^p(B(0,R)) \int_{\mathbb{B}} \omega(z)d\mu_{Q+pt}(z) < \infty,$$

then

$$\int_{\mathbb{B}} \omega(z)d\mu_{Q+pt}(z) < \infty,$$

and $\omega \in L^1(d\mu_{Q+pt})$. □

Lemma 7.4.3. *Let ω be a positive locally integrable function and $q, Q \in \mathbb{R}$. If $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, then $\omega^{\frac{-1}{p-1}} \in L^1(d\mu_{q+p'(b-q)})$.*

Proof. Assume that $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. We want to show that $\omega^{\frac{-1}{p-1}} \in L^1(d\mu_{q+p(p'-1)(b-q)})$, in other words we want to show $\omega^{-1} \in L^{p'}(\omega d\mu_{q+p(p'-1)(b-q)})$. Assume that ω^{-1} is not in $L^{p'}(\omega d\mu_{q+p(p'-1)(b-q)})$, then by the Riesz representation theorem there exists a positive function h in

$L^p(\omega d\mu_{q+p(p'-1)(b-q)})$ such that

$$\langle h, \omega^{-1}\rangle_{\omega,q+p(p'-1)(b-q)} = \infty.$$

This means that

$$\begin{aligned}
\infty &= \int_{\mathbb{B}} h(z) d\mu_{q+p(p'-1)(b-q)}(z) \\
&= \int_{\mathbb{B}} h(z)(1-|z|^2)^{p(p'-1)(b-q)} d\mu_q(z) \\
&= \int_{\mathbb{B}} h(z)(1-|z|^2)^{[p(p'-1)-1](b-q)} d\mu_b(z) \\
&= \int_{\mathbb{B}} h(z)(1-|z|^2)^{[p'-1](b-q)} d\mu_b(z).
\end{aligned}$$

Since $h \in L^p(\omega d\mu_{q+p(p'-1)(b-q)})$, then $g \in L^p(\omega d\mu_q)$ where $g(z) = h(z)(1-|z|^2)^{(p'-1)(b-q)}$, $\forall z \in \mathbb{B}$. So that

$$T_{a,b}g(0) = \int_{\mathbb{B}} h(z)(1-|z|^2)^{[p'-1](b-q)} d\mu_b(z) = \infty,$$

contradicting the fact that $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. $\square$

Lemma 7.4.4. *Let ω be a positive locally integrable function and $q, Q \in \mathbb{R}$. If $P_{s,t}$ ($s, t \in \mathbb{R}$) is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, then $\omega^{\frac{-1}{p-1}} \in L^1(d\mu_{q+p'(s-q)})$.*

Proof. For the proof, we refer to the proof of Lemma 7.4.3 and notice that $P_{s,t}g(0) = T_{s+t,s}g(0)$. $\square$

Proposition 7.4.5. *In the case $s+t \leq -1$ and $Q \leq q$, or in the case $s+t+\frac{Q-q}{p} \leq -1$, there are no weights ω such that both conditions $\omega \in L^1(d\mu_{Q+pt})$ and $\omega^{\frac{-1}{p-1}} \in L^1(d\mu_{q+p'(s-q)})$ hold at the same time.*

Proof. Let $s+t \leq -1$, $Q \leq q$ and B be a pseudo-ball that touches the boundary. Then

$$\begin{aligned}
\infty &= \int_B d\mu_{s+t}(z) \\
&= \int_B (1-|z|^2)^{s+t} d\mu(z) \\
&= \int_B (1-|z|^2)^{s-\frac{q}{p}+\frac{q}{p}+t} d\mu(z) \\
&= \int_B \omega^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{q+pt}{p}} \omega^{-\frac{1}{p}}(z)(1-|z|^2)^{s-\frac{q}{p}} d\mu(z) \\
&\leq \left(\int_B \omega(z)(1-|z|^2)^{q+pt} d\mu(z)\right)^{\frac{1}{p}} \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z)\right)^{\frac{1}{p'}} \\
&\leq \left(\int_B \omega(z)(1-|z|^2)^{Q+pt} d\mu(z)\right)^{\frac{1}{p}} \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z)\right)^{\frac{1}{p'}}
\end{aligned}$$

such that necessarily $\omega \notin L^1(d\mu_{Q+pt})$ or $\omega^{\frac{-1}{p-1}} \notin L^1(d\mu_{q+p'(s-q)})$.

Let $s+t+\frac{Q-q}{p} \leq -1$, and B be a pseudo-ball that touches the boundary. Then

$$\begin{aligned}
\infty &= \int_B d\mu_{s+t+\frac{Q-q}{p}}(z) \\
&= \int_B (1-|z|^2)^{s+t+\frac{Q-q}{p}} d\mu(z) \\
&= \int_B (1-|z|^2)^{s-\frac{q}{p}+\frac{Q}{p}+t} d\mu(z) \\
&= \int_B \omega^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{Q+pt}{p}} \omega^{-\frac{1}{p}}(z)(1-|z|^2)^{s-\frac{q}{p}} d\mu(z) \\
&\leq \left(\int_B \omega(z)(1-|z|^2)^{Q+pt} d\mu(z)\right)^{\frac{1}{p}} \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z)\right)^{\frac{1}{p'}}
\end{aligned}$$

such that necessarily $\omega \notin L^1(d\mu_{Q+pt})$ or $\omega^{\frac{-1}{p-1}} \notin L^1(d\mu_{q+p'(s-q)})$.

□

7.4.2 The case where $a < -N-1$ and $s+t < -N-1$

Here we are going to characterize the boundedness of the operators $T_{a,b}$, $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ where ω is a positive locally integrable function, where $a < -N-1$ and $s+t < -N-1$.

Theorem 7.4.6. *In the case $a < -(N+1)$, $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ if and only if*

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_Q(z)\right)\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z)\right)^{p-1} < \infty.$$

Moreover,

$$\|T_{a,b}\|^p \simeq \left(\int_{\mathbb{B}} \omega(z) d\mu_Q(z)\right)\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z)\right)^{p-1}.$$

Proof. Assume that

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_Q(z)\right)\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z)\right)^{p-1} < \infty.$$

We have

$$\begin{aligned}\left(\int_{\mathbb{B}} |T_{a,b}f(z)|^p \omega(z) d\mu_Q(z)\right) &\lesssim \int_{\mathbb{B}} \left(\int_{\mathbb{B}} |f(v)| d\mu_b(v)\right)^p \omega(z) d\mu_Q(z)\\ &= \left(\int_{\mathbb{B}} \omega(z) d\mu_Q(z)\right)\left(\int_{\mathbb{B}} |f(z)| d\mu_b(z)\right)^p,\end{aligned}$$

where the first inequality is due to the fact that K_a is bounded when $a < -(N+1)$. We also have

$$\begin{aligned}\left(\int_{\mathbb{B}} |f(z)| d\mu_b(z)\right)^p &= \left(\int_{\mathbb{B}} |f(z)|(1-|z|^2)^{b-q} d\mu_q(z)\right)^p\\ &= \left(\int_{\mathbb{B}} |f(z)|(\omega(z))^{\frac{-1}{p}}(\omega(z))^{\frac{1}{p}}(1-|z|^2)^{b-q} d\mu_q(z)\right)^p\\ &\leq \left(\int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z)\right)\left(\int_{\mathbb{B}} ((\omega(z))^{\frac{-1}{p}})^{p'}|(1-|z|^2)^{p'(b-q)} d\mu\end{aligned}$$

Then $T_{a,b}$ is well defined and continuous in $L^p(\omega d\mu_q)$ when

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_Q(z)\right)\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z)\right)^{p-1} < \infty.$$

Now assume that $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to

$L^p(\omega d\mu_Q)$. Let ρ_0 be as in Lemma 7.2.4, then for positive functions we have

$$\begin{aligned}
\int_{\mathbb{B}} |T_{a,b}f(z)|^p \omega(z) d\mu_Q(z) &= \int_{\mathbb{B}} |\int_{\mathbb{B}} K_a(z,s) f(s) d\mu_b(s)|^p \omega(z) d\mu_Q(z) \\
&\geq \int_{\mathbb{B}} |\int_{\mathbb{B}} \Re K_a(z,s) f(s) d\mu_b(s)|^p \omega(z) d\mu_Q(z) \\
&\geq \int_{|z|\leq\rho_0} |\int_{\mathbb{B}} \Re K_a(z,s) f(s) d\mu_b(s)|^p \omega(z) d\mu_Q(z) \\
&\geq \frac{1}{2^p} \int_{|z|\leq\rho_0} (\int_{\mathbb{B}} |f(s)| d\mu_b(s))^p \omega(z) d\mu_Q(z) \\
&= \frac{1}{2^p} (\int_{\mathbb{B}} |f(s)| d\mu_b(s))^p \int_{|z|\leq\rho_0} \omega(z) d\mu_Q(z).
\end{aligned}$$

By continuity of $T_{a,b}$ there exists a constant $C_{a,b,p,q,Q} > 0$ such that

$$\int_{\mathbb{B}} |T_{a,b}f(z)|^p \omega(z) d\mu_Q(z) \leq C_{a,b,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z),$$

for all f in $L^p(\omega d\mu_q)$. Hence,

$$\frac{1}{2^p} \left(\int_{|z|\leq\rho_0} \omega(z) d\mu_Q(z) \right) \left(\int_{\mathbb{B}} |f(z)| d\mu_b(z) \right)^p \leq C_{a,b,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z)$$

for all positive f in $L^p(\omega d\mu_q)$. Let $f(z) = (\omega(z))^{\frac{-1}{p-1}} (1-|z|^2)^{(p'-1)(b-q)}, \quad z \in \mathbb{B}$.

We have that $f \in L^p(\omega d\mu_q)$ by Lemma 7.4.3. Replacing f in the last inequality we obtain

$$\left(\int_{|z|\leq\rho_0} \omega(z) d\mu_Q(z) \right) \left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z) \right)^{p-1} < 2^p C_{a,b,p,q,Q} < \infty.$$

Then

$$\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z) \right)^{p-1} < \infty.$$

Using Lemma 7.4.1, we get

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_Q(z) \right) \left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z) \right)^{p-1} < \infty.$$

□

Theorem 7.4.7. *In the case $s+t < -(N+1)$, there are no weights ω such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \leq q$.*

Proof. This is due to Proposition 7.4.5, Lemma 7.4.2 and Lemma 7.4.4. $\square$

Theorem 7.4.8. *In the case $s+t < -(N+1)$, if $Q > q$, then $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ if and only if*

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty.$$

Moreover,

$$\|P_{s,t}\|^p \simeq \left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1}.$$

Proof. Assume that

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty.$$

We have

$$\begin{aligned}
\left(\int_{\mathbb{B}} |P_{s,t}f(z)|^p \omega(z) d\mu_Q(z)\right) &\lesssim \int_{\mathbb{B}} (1-|z|^2)^{pt} \left(\int_{\mathbb{B}} |f(v)| d\mu_s(v)\right)^p \omega(z) d\mu_Q(z) \\
&= \int_{\mathbb{B}} \left(\int_{\mathbb{B}} |f(v)| d\mu_s(v)\right)^p \omega(z) d\mu_{Q+pt}(z) \\
&= \left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}} |f(z)| d\mu_s(z)\right)^p,
\end{aligned}$$

where the first inequality is due to the fact that K_{s+t} is bounded when $s+t < -(N+1)$. We also have

$$\begin{aligned}
\left(\int_{\mathbb{B}} |f(z)| d\mu_s(z)\right)^p &= \left(\int_{\mathbb{B}} |f(z)|(\omega(z))^{\frac{-1}{p}}(\omega(z))^{\frac{1}{p}}(1-|z|^2)^{s-q} d\mu_q(z)\right)^p \\
&\leq \left(\int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z)\right)\left(\int_{\mathbb{B}} ((\omega(z))^{\frac{-1}{p}})^{p'} |(1-|z|^2)^{p'(s-q)} d\mu
\end{aligned}$$

Then $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ when

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty.$$

Now assume that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to

$L^p(\omega d\mu_Q)$. Let ρ_0 be as in Lemma 7.2.4, then for positive functions f we have

$$\int_{\mathbb{B}} |P_{s,t}f(z)|^p \omega(z) d\mu_Q(z) = \int_{\mathbb{B}} | \int_{\mathbb{B}} K_{s+t}(z,w) f(w) d\mu_s(w)|^p \omega(z) d\mu_{Q+pt}(z)$$
$$\geq \int_{|z|\leq\rho_0} | \int_{\mathbb{B}} \Re K_{s+t}(z,w) f(w) d\mu_s(w)|^p \omega(z) d\mu_{Q+pt}(z)$$
$$= \frac{1}{2^p} (\int_{\mathbb{B}} f(w) d\mu_s(w))^p \int_{|z|\leq\rho_0} \omega(z) d\mu_{Q+pt}(z).$$

By continuity of $P_{s,t}$ there exists a constant $C_{s,t,p,q,Q} > 0$ such that

$$\int_{\mathbb{B}} |P_{s,t}f(z)|^p \omega(z) d\mu_Q(z) \leq C_{s,t,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z),$$

for all f in $L^p(\omega d\mu_q)$. Hence,

$$\frac{1}{2^p} \left(\int_{|z|\leq\rho_0} \omega(z) d\mu_{Q+pt}(z) \right) \left(\int_{\mathbb{B}} |f(z)| d\mu_s(z) \right)^p \leq C_{s,t,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z)$$

for all positive f in $L^p(\omega d\mu_q)$. Let $f(z) = (\omega(z))^{\frac{-1}{p-1}} (1-|z|^2)^{(p'-1)(s-q)}$, $\forall z \in \mathbb{B}$. Then $f \in L^p(\omega d\mu_q)$ by Lemma 7.4.4; Replacing f in the last inequality we obtain

$$\left(\int_{|z|\leq\rho_0} \omega(z) d\mu_{Q+pt}(z) \right) \left(\int_{\mathbb{B}} (\omega(z)^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < 2^p C_{s,t,p,q,Q} < \infty.$$

Then

$$\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty.$$

Using Lemma 7.4.2, we get

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z) \right) \left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty.$$

□

7.5 The case where $a > -N-1$ and $s+t > -N-1$

Here we are going to start the study of the boundedness of the operators $T_{a,b}$ and $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ in the case $a > -(N+1)$ and $s+t > -(N+1)$

respectively.

7.5.1 Necessary Conditions

To obtain our necessaries conditions, we are going to use the following lemma.

Lemma 7.5.1. *Let B_i be a pseudo-ball of radius R sufficiently small touching the boundary of $\mathbb{B}$. There is a pseudo-ball B_j with the same radius touching the boundary of $\mathbb{B}$, sufficiently far from B_i but such that, if w_0 is the center of B_i: $d(z, w_0) \geq C_1 d(w, w_0)$, for every $z \in B_j$ and for every $w \in B_i$, for C_1 as in Lemma 7.2.15. For such B_j, for all non negative functions with support in B_i we have if $a > -1 - N$:*

$$|T_{a,b} f(z)| \geq C_{a,b} \frac{1}{R^{N+1+a}} \int_{B_i} f(w) d\mu_b(w).$$

In particular, if $a > -1$ then:

$$|T_{a,b} f(z)| \geq C_{a,b} \frac{1}{\mu_a(B_i)} \int_{B_i} f(w) d\mu_b(w)$$

for all $z \in B_j$ $i \neq j; i, j = 1, 2$. The constant $C_{a,b}$ does not depend of B_i, B_j and f. Then if $T_{a,b}$ is well defined, then b has to be larger than -1.

Proof. Let w_0 be the center of B_i. If R is sufficiently small, we take B_j such that for all z in B_j and for all w in B_i, we have $d(z, w_0) \geq C_1 d(w, w_0)$ where C_1 is as in Lemma 7.2.15. Let f be a non negative function with support in B_i and let $z \in B_j$, we have

$$\begin{aligned} T_{a,b} f(z) = & \frac{1}{(1 - \langle z, w_0 \rangle)^{N+1+a}} \int_{B_i} f(w) d\mu_b(w) \\ & + \int_{B_i} \left[\frac{1}{(1 - \langle z, w \rangle)^{N+1+a}} - \frac{1}{(1 - \langle z, w_0 \rangle)^{N+1+a}} \right] f(w) d\mu_b(w). \end{aligned}$$

Then

$$|T_{a,b}f(z)| \geq \frac{1}{|1-\langle z, w_0\rangle|^{N+1+a}} \int_{B_i} f(w)d\mu_b(w)$$
$$- \int_{B_i} \left| \frac{1}{(1-\langle z, w\rangle)^{N+1+a}} - \frac{1}{(1-\langle z, w_0\rangle)^{N+1+a}} \right| f(w)d\mu_b(w)$$

By Lemma 7.2.15 and Proposition 7.2.2 we have

$$\left| \frac{1}{(1-\langle z, w\rangle)^{N+1+a}} - \frac{1}{(1-\langle z, w_0\rangle)^{N+1+a}} \right| \leq \frac{1}{2} \frac{1}{|1-\langle z, w_0\rangle|^{N+a+1}},$$

so that

$$|T_{a,b}f(z)| \geq \frac{1}{2} \frac{1}{|1-\langle z, w_0\rangle|^{N+1+a}} \int_{B_i} f(w)d\mu_b(w).$$

Since our pseudo-balls touch the boundary and since $d(B_i, B_j) \simeq R$, we have

$$|1-\langle z, w_0\rangle| \lesssim R.$$

Hence

$$|T_{a,b}f(z)| \gtrsim \frac{1}{2} \frac{1}{R^{N+1+a}} \int_{B_i} f(w)d\mu_b(w).$$

By Lemma 7.2.3, if $a > -1$ and because B_i touches the boundary we have $\mu_a(B_i) \simeq R^{N+1+a}$, so that

$$|T_{a,b}f(z)| \gtrsim \frac{1}{2} \frac{1}{\mu_a(B_i)} \int_{B_i} f(w)d\mu_b(w).$$

□

Lemma 7.5.2. *Let B_i be a pseudo-ball of radius R sufficiently small touching the boundary of $\mathbb{B}$. There is a pseudo-ball B_j with the same radius touching the boundary of $\mathbb{B}$, sufficiently far from B_i but such that, if w_0 is the center of B_i: $d(z, w_0) \geq C_1 d(w, w_0)$, for every $z \in B_j$ and for every $w \in B_i$, for C_1 as in Lemma 7.2.15. For such B_j, for all non negative functions with support in B_i; we have when $-1-N < s+t$:*

$$|P_{s,t}f(z)| \geq C_{s,t} \frac{(1-|z|^2)^t}{R^{N+1+s+t}} \int_{B_i} f(w)d\mu_s(w),$$

and when $s+t>-1$ we get:

$$|P_{s,t}f(z)| \geq C_{s,t}\frac{(1-|z|^2)^t}{\mu_{s+t}(B_i)}\int_{B_i} f(w)d\mu_s(w)$$

for all $z \in B_j$ $i \neq j; i,j = 1,2$. The constant $C_{s,t}$ does not depend of B_i, B_j and f. Then if $P_{s,t}$ is well defined, s has to be larger than -1.

Proof. This is the consequence of Lemma 7.5.1 because $P_{s,t}f(z) = (1-|z|^2)^t T_{s+t,s}f$

□

We are now ready to prove Theorem 7.1.9.

Theorem 7.5.3. *If $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \leq q$ then if $a > -1$ we have:*

$$\sup_{pseudo-balls\ B: B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{\mu_b(B)}{\mu_a^2(B)}\int_B \omega(z)d\mu_Q(z)\right)\left(\frac{\mu_b(B)}{\mu_a^2(B)}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z$$

∞.

More generally if $a > -(N+1)$, then:

$$\sup_{pseudo-balls\ B: B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{\mu_b(B)}{R_B^{2(N+1+a)}}\int_B \omega(z)d\mu_Q(z)\right)\left(\frac{\mu_b(B)}{R_B^{2(N+1+a)}}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q}$$

∞

where R_B is the radius of B.

Proof. Assume that $a > -(N+1)$ and $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \leq q$. Then there exists a constant $C_{a,b,p,q,Q} > 0$ such that

$$\int_{\mathbb{B}} |T_{a,b}f(z)|^p\omega(z)d\mu_Q(z) \leq C_{a,b,p,q,Q}\int_{\mathbb{B}} |f(z)|^p\omega(z)d\mu_q(z).$$

Let f be a positive function with support in B_i (we take B_i, B_j as in Lemma 7.5.1). By Lemma 7.5.1, we then have in the case $a > -1$:

$$C_{a,b}^p\int_{B_j}\frac{1}{\mu_a^p(B_i)}\left(\int_{B_i} f(w)d\mu_b(w)\right)^p\omega(z)d\mu_Q(z) \leq C_{a,b,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z$$

hence

$$\frac{1}{\mu_a^p(B_i)}\left(\int_{B_i} f(w)d\mu_b(w)\right)^p\left(\int_{B_j}\omega(z)d\mu_Q(z)\right) \leq C'_{a,b,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z)$$

$$\leq C'_{a,b,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_Q(z)$$

Choosing $f = \frac{\mu_a(B_i)}{\mu_b(B_i)}\chi_{B_i}$ in the last inequality we get

$$\omega(B_j) \leq C'_{a,b,p,q,Q}\frac{\mu_a^p(B_i)}{\mu_b^p(B_i)}\omega(B_i),$$

where $\omega(B_k) = \displaystyle\int_{B_k}\omega(z)d\mu_Q(z)$, $k = i, j$.

As B_i and B_j touch the boundary of $\mathbb{B}$ and have the same radius, by Lemma 7.2.3 we have

$$\frac{\mu_b^a(B_i)}{\mu_b^p(B_i)} \simeq \frac{\mu_a^p(B_j)}{\mu_b^p(B_j)}.$$

We then have

$$\omega(B_j) \leq C''_{a,b,p,q,Q}\frac{\mu_a^p(B_j)}{\mu_b^p(B_j)}\omega(B_i).$$

Interchanging B_i and B_j (See Lemma 7.5.1) we get

$$\omega(B_i) \leq C''_{a,b,p,q,Q}\frac{\mu_a^p(B_i)}{\mu_b^p(B_i)}\omega(B_j),$$

so that

$$\frac{\mu_b^p(B_i)}{\mu_a^p(B_i)}\omega(B_i) \leq C''_{a,b,p,q,Q}\omega(B_j)$$

which together with

$$\frac{1}{\mu_a^p(B_i)}\left(\int_{B_i} f(w)d\mu_b(w)\right)^p\left(\int_{B_j}\omega(z)d\mu_Q(z)\right) \leq C'_{a,b,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z)$$

leads to

$$\frac{\mu_b^p(B_i)}{\mu_a^{2p}(B_i)}\left(\int_{B_i} f(w)d\mu_b(w)\right)^p\left(\int_{B_i}\omega(z)d\mu_Q(z)\right) \leq C'''_{a,b,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z)$$

Then choosing $f(z) = \omega^{\frac{-1}{p-1}}(z)(1-|z|^2)^{(p'-1)(b-q)}\chi_{B_i}(z)$ ($f \in L^p(\omega d\mu_q)$ by Lemma 7.4.3) in that last inequality we obtain

$$\left(\frac{\mu_b(B_i)}{\mu_a^2(B_i)}\int_{B_i}\omega(z)d\mu_Q(z)\right)\left(\frac{\mu_b(B_i)}{\mu_a^2(B_i)}\int_{B_i}(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z)\right)^{p-1} \leq C'''_{a,b,p,q,Q}$$

when $-1-N < a < -1$ it is sufficient to replace $\mu_a(B_i)$ by R^{N+1+a} in the proof. □

Theorem 7.5.4. *If $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q > q$ then if $a > -1$ we have:*

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B\omega(z)d\mu_Q(z)\right)\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z)\right.$$

where the supremum is taken over the pseudoballs B.

More generally if $a > -(N+1)$, then:

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R_B^{2(N+1+a)}}\int_B\omega(z)d\mu_Q(z)\right)\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R_B^{2(N+1+a)}}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z)\right.$$

where the supremum is taken over the pseudoballs B with radius R_B.

Proof. The proof is similar to the proof of Theorem 7.5.3, except that we choose the first testing function to be $f(z) = (1-|z|^2)^{\frac{Q-q}{p}}\chi_{B_i}$. □

Theorem 7.5.5. *In the case $-(N+1) < s+t < -1$ and $Q \leq q$, or in the case $s+t+\frac{Q-q}{p} \leq -1$, there are no weights ω such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$.*

Proof. This is due to Proposition 7.4.5, Lemma 7.4.2 and Lemma 7.4.4. □

Theorem 7.5.6. *In the case $-(N+1) < s+t < -1$, with $s+t+\frac{Q-q}{p} > -1$ if $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ then*

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{R_B^{\frac{Q-q}{p}}}{R_B^{N+1+s+t}}\int_B\omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{R_B^{\frac{Q-q}{p}}}{R_B^{N+1+s+t}}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}\right.$$

where the supremum is taken over the pseudoballs B with radius R_B.

Proof. Assume that $-1 > s+t > -(N+1)$, with $s+t+\frac{Q-q}{p} > -1$ and $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. Then there exists a constant $C_{s,t,p,q,Q} > 0$ such that

$$\int_{\mathbb{B}} |P_{s,t}f(z)|^p \omega(z) d\mu_Q(z) \le C_{s,t,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z).$$

Let f be a positive function with support in B_i (we take B_i, B_j of radius R as in Lemma 7.5.2). By Lemma 7.5.2, we then have

$$C^p_{s,t} \int_{B_j} \frac{1}{R^{(N+1+s+t)p}} \left(\int_{B_i} f(w) d\mu_s(w) \right)^p \omega(z) d\mu_{Q+pt}(z) \le C_{s,t,p,q} \int_{B_i} |f(z)|^p \omega(z$$

hence

$$\frac{1}{R^{(N+1+s+t)p}} \left(\int_{B_i} f(w) d\mu_s(w) \right)^p \left(\int_{B_j} \omega(z) d\mu_{Q+pt}(z) \right) \le C'_{s,t,p,q} \int_{B_i} |f(z)|^p \omega(z)$$

Choosing $f(z) = (1-|z|^2)^{\frac{Q-q}{p}+t} \chi_{B_i}(z)$ in the last inequality we get, because $N+s+t > -1$,

$$R^{Q-q} \int_{B_j} \omega(z) d\mu_{Q+pt}(z) \le \int_{B_i} \omega(z) d\mu_{Q+pt}(z).$$

Interchanging B_i and B_j (See Lemma 7.5.2) we get

$$R^{Q-q} \int_{B_i} \omega(z) d\mu_{Q+pt}(z) \lesssim \int_{B_j} \omega(z) d\mu_{Q+pt}(z),$$

which together with

$$\frac{1}{R^{p(N+1+s+t)}} \left(\int_{B_i} f(w) d\mu_s(w) \right)^p \left(\int_{B_j} \omega(z) d\mu_{Q+pt}(z) \right) \le C'_{s,t,p,q,Q} \int_{B_i} |f(z)|^p \omega($$

lead to

$$\frac{R^{Q-q}}{R^{p(N+1+s+t)}} \left(\int_{B_i} f(w) d\mu_s(w) \right)^p \left(\int_{B_i} \omega(z) d\mu_{Q+pt}(z) \right) \le C'_{s,t,p,q,Q} \int_{B_i} |f(z)|^p \omega($$

Then choosing $f(z) = \omega^{\frac{-1}{p-1}}(z)(1-|z|^2)^{(p'-1)(s-q)} \chi_{B_i}(z)$ ($f \in L^p(\omega d\mu_q)$ by Lemma 7.4.4) in that last inequality, we obtain

$$\left(\frac{R^{\frac{Q-q}{p}}}{R^{N+1+s+t}} \int_{B_i} \omega(z) d\mu_{Q+pt}(z) \right) \left(\frac{R^{\frac{Q-q}{p}}}{R^{N+1+s+t}} \int_{B_i} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} \le$$

$\square$

In the same way we have

Theorem 7.5.7. *If $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \geq q$, then if $s+t > -1$ we have:*

$$\sup_{pseudo-balls\ B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{R_B^{\frac{Q-q}{p}}}{\mu_{s+t}(B)} \int_B \omega(z) d\mu_{Q+pt}(z) \right) \left(\frac{R_B^{\frac{Q-q}{p}}}{\mu_{s+t}(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+}$$

where R_B is the radius of B.

Proof. The proof is similar to the proof of Theorem 7.5.6, we choose the first testing function to be $f(z) = (1-|z|^2)^{\frac{Q-q}{p}+t} \chi_{B_i}$. $\square$

Theorem 7.5.8. *In the case $s+t > -1$ there are no weights ω such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q < q$.*

Proof. We are going to proceed in two steps.

Step 1: Show that in the case $s+t > -1$, if $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, for $Q < q$, then we have:

$$\sup_{pseudo-balls\ B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{1}{\mu_{s+t}(B)} \int_B \omega(z) d\mu_{Q+pt}(z) \right) \left(\frac{1}{\mu_{s+t}(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p}$$

Assume that $s+t > -1$ and $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q < q$. There exists a constant $C_{s,t,p,q,Q} > 0$ such that

$$\int_{\mathbb{B}} |P_{s,t} f(z)|^p \omega(z) d\mu_Q(z) \leq C_{s,t,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z), \quad f \in L^p(\omega d\mu_q).$$

Let f be a positive function with support in B_i (we take B_i, B_j as in Lemma 7.5.2). By Lemma 7.5.2, we then have

$$\frac{1}{\mu_{s+t}^p(B_i)}\left(\int_{B_i} f(w)d\mu_s(w)\right)^p\left(\int_{B_j}\omega(z)d\mu_{Q+pt}(z)\right) \le C'_{s,t,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu$$

$$\le C'_{s,t,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu$$

Choosing $f(z) = (1-|z|^2)^t\chi_{B_i}(z)$ in the last inequality we get

$$\int_{B_j}\omega(z)d\mu_{Q+pt}(z) \le \int_{B_i}\omega(z)d\mu_{Q+pt}(z).$$

Interchanging B_i and B_j (see Lemma 7.5.2) we get

$$\int_{B_i}\omega(z)d\mu_{Q+pt}(z) \le \int_{B_j}\omega(z)d\mu_{Q+pt}(z),$$

which together with

$$\frac{1}{\mu_{s+t}^p(B_i)}\left(\int_{B_i} f(w)d\mu_s(w)\right)^p\left(\int_{B_j}\omega(z)d\mu_{Q+pt}(z)\right) \le C'_{s,t,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu$$

lead to

$$\frac{1}{\mu_{s+t}^p(B_i)}\left(\int_{B_i} f(w)d\mu_s(w)\right)^p\left(\int_{B_i}\omega(z)d\mu_{Q+pt}(z)\right) \le C'_{s,t,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu$$

Then choosing $f(z) = (\omega^{\frac{-1}{p-1}}(z))(1-|z|^2)^{(p'-1)(s-q)}\chi_{B_i}(z)$ ($f \in L^p(\omega d\mu_q)$ by Lemma 7.4.4) in that last inequality we obtain

$$\left(\frac{1}{\mu_{s+t}(B)}\int_B\omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t}(B)}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1} \le C'''_{s,t,p,}$$

Step 2: Show that

$$\sup_{pseudo-balls\ B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{1}{\mu_{s+t}(B)}\int_B\omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t}(B)}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'}\right.$$

Let

$$II = \left(\frac{1}{\mu_{s+t}(B)}\int_B\omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t}(B)}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}.$$

Let B be a pseudo-ball that touches the boundary and R_B its radius. By Hölder's inequality we have

$$
\begin{aligned}
\mu_{s+t}(B) &= \int_B d\mu_{s+t}(z) \\
&= \int_B (1-|z|^2)^{s+t} d\mu(z) \\
&= \int_B (1-|z|^2)^{s-\frac{q}{p}+\frac{q}{p}+t} d\mu(z) \\
&= \int_B \omega^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{q+pt}{p}} \omega^{-\frac{1}{p}}(z)(1-|z|^2)^{s-\frac{q}{q}} d\mu(z) \\
&\leq \left(\int_B \omega(z)(1-|z|^2)^{q+pt} d\mu(z)\right)^{\frac{1}{p}} \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z)\right)
\end{aligned}
$$

Note that for $z \in B$ we have $1-|z| < 2R_B$. Then

$$
\begin{aligned}
\mu_{s+t}^p(B) &\leq \left(\int_B \omega(z)(1-|z|^2)^{q+pt} d\mu(z)\right) \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z)\right)^{p} \\
&= \left(\int_B \omega(z)(1-|z|^2)^{q-Q+Q+pt} d\mu(z)\right) \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu \right. \\
&\leq (2R_B)^{q-Q} \left(\int_B \omega(z)(1-|z|^2)^{Q+pt} d\mu(z)\right) \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s} \right.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(2R_B)^{Q-q} &\leq \frac{1}{\mu_{s+t}^p(B)} \left(\int_B \omega(z)(1-|z|^2)^{Q+pt} d\mu(z)\right) \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s} \right. \\
&= II.
\end{aligned}
$$

Taking the sup over smaller radii, we get $\sup II = \infty$. $\square$

7.5.2 Maximal Functions

We introduce for $b > -1$ and $s > -1$ the following maximal functions. If $a > -1$ we set:

$$
m_{a,b} f(z) = \sup_{\zeta \in \mathbb{B}, R > 1-|\zeta| : z \in B(\zeta,R)} \frac{1}{\mu_a(B(\zeta,R))} \int_{B(\zeta,R)} |f(w)| d\mu_b(w), \quad (7.5.2.1)
$$

more generally if $a > -1 - N$ we set:

$$m'_{a,b} f(z) = \sup_{\zeta \in \mathbb{B}, R > 1-|\zeta| : z \in B(\zeta, R)} \frac{1}{R^{N+1+a}} \int_{B(\zeta,R)} |f(w)| d\mu_b(w). \tag{7.5.2.2}$$

If $a > -1$ we set:

$$M_{a,b} f(z) = \sup_{B : z \in B} \frac{1}{\mu_a(B)} \int_B |f(w)| d\mu_b(w), \tag{7.5.2.3}$$

and more generally if $a > -1 - N$ we set:

$$M'_{a,b} f(z) = \sup_{B : z \in B} \frac{1}{R^{N+1+a}} \int_B |f(w)| d\mu_b(w). \tag{7.5.2.4}$$

If $s + t > -1$ we set:

$$O_{s,t} f(z) = (1-|z|^2)^t \sup_{\zeta \in \mathbb{B}, R > 1-|\zeta| : z \in B(\zeta, R)} \frac{1}{\mu_{s+t}(B(\zeta, R))} \int_{B(\zeta,R)} |f(w)| d\mu_s(w), \tag{7.5.2.5}$$

more generally if $s + t > -1 - N$ we set:

$$O'_{s,t} f(z) = (1-|z|^2)^t \sup_{\zeta \in \mathbb{B}, R > 1-|\zeta| : z \in B(\zeta, R)} \frac{1}{R^{N+1+s+t}} \int_{B(\zeta,R)} |f(w)| d\mu_s(w). \tag{7.5.2.6}$$

Let finally define the following fractional maximal function

$$M_\gamma f(z) = \sup_{B : z \in B} \frac{1}{\mu_b^{1-\gamma}(B)} \int_B |f(w)| d\mu_b(w), \quad \gamma \in (0,1). \tag{7.5.2.7}$$

Note that for $a < b$ we get by Lemma 7.2.3 $M_{a,b} \sim M_\gamma$ with $\gamma = 1 - \frac{N+1+a}{N+1+b}$.

For all $k \in (0,1)$, we define the operator of regularisation R_k^b

$$R_k^b f(z) = \frac{1}{\mu_b(B_k(z))} \int_{B_k(z)} f(\zeta) d\mu_b(\zeta), \tag{7.5.2.8}$$

where $B_k(z) = \{w \in \mathbb{B} : d(z,w) < k(1-|z|)\}$.

We will need the following lemmas to show Theorem 7.1.16. See [104] for the proofs of the first two lemmas.

Lemma 7.5.9. *Let $k \in (0, \frac{1}{2})$. If $z' \in B_k(z)$, then $z \in B_{k'}(z')$, where $k' = \frac{k}{1-k}$.*

Lemma 7.5.10. *If $B := B(x, R)$ touches the boundary then if we take $B' = B(x, K(1+2k_1)R)$, then $\forall w \in B, B_{k_1}(w) \subset B'$.*

Lemma 7.5.11. *For all $k \in (0,1)$, there is a constant C_k such that for all positive locally integrable function f we have if $a > -1$:*

$$m_{a,b} f \leq C_k m_{a,b}(R_k^b f),$$

and more generally if $a > -1 - N$:

$$m'_{a,b} f \leq C_k m'_{a,b}(R_k^b f).$$

Proof. We have to show that for all z and all pseudo-balls B containing z which touch the boundary of $\mathbb{B}$, there is a pseudo-ball $z \in B'$ which touches the boundary of $\mathbb{B}$ so that

$$\frac{1}{\mu_a(B)} \int_B f(w) d\mu_b(w) \leq C_k \frac{1}{\mu_a(B')} \int_{B'} \left[\frac{1}{\mu_b(B_k(w))} \int_{B_k(w)} f(\zeta) d\mu_b(\zeta) \right] d\mu_b(w).$$

By Lemma 7.5.9, $\chi_{B_k(w)}(\zeta) \geq \chi_{B_{k_1}(\zeta)}(w)$, where $k_1 = \frac{k}{k+1}$. If $B = B(x, R)$ ($R > 1 - |x|$) by Lemma 7.5.10 if $B' = B(x, K(1+2k_1)R)$, then $\forall w \in B, B_{k_1}(w) \subset B'$. Note that

$$\mu_b(B_{k_1}(\zeta)) \simeq \mu_b(B_k(w)) \quad \text{when} \quad \mathrm{w} \in \mathrm{B}_{\mathrm{k}_1}(\zeta). \tag{7.5.2.9}$$

Hence

$$\begin{aligned}
\int_{B'} R_k^b f(w) d\mu_b(w) &= \int_{B'} \left[\frac{1}{\mu_b(B_k(w))} \int_{B_k(w)} f(\zeta) d\mu_b(\zeta) \right] d\mu_b(w) \\
&= \int_{B'} \left[\frac{1}{\mu_b(B_k(w))} \int_B f(\zeta) \chi_{B_k(w)}(\zeta) d\mu_b(\zeta) \right] d\mu_b(w) \\
&\geq \int_{B'} \left[\frac{1}{\mu_b(B_k(w))} \int_B f(\zeta) \chi_{B_{k_1}(\zeta)}(w) d\mu_b(\zeta) \right] d\mu_b(w) \\
&= \int_B \left[\int_{B'} \frac{1}{\mu_b(B_k(w))} \chi_{B_{k_1}(\zeta)}(w) d\mu_b(w) \right] f(\zeta) d\mu_b(\zeta).
\end{aligned}$$

Using (7.5.2.9), we get

$$\int_{B'} R_k^b f(w) d\mu_b(w) \gtrsim \int_B \frac{1}{\mu_b(B_{k_1}(\zeta))} \left[\int_{B'} \chi_{B_{k_1}(\zeta)}(w) d\mu_b(w)\right] f(\zeta) d\mu_b(\zeta)$$
$$= \int_B f(\zeta) d\mu_b(\zeta).$$

Since μ_a is a homogeneous measure we have

$$\frac{1}{\mu_a(B)} \int_B f(w) d\mu_b(w) \lesssim \frac{1}{\mu_a(B')} \int_{B'} R_k^b f(w) d\mu_b(w).$$

For $m'_{a,b}$ it is sufficient to observe that B and B' have equivalent radii. $\square$

The following lemma appears as a corollary of the preceding one by observing that

$$O_{s,t} f(z) := (1 - |z|^2)^t m_{s+t,s} f(z)$$

and

$$O'_{s,t} f(z) := (1 - |z|^2)^t m'_{s+t,s} f(z).$$

Lemma 7.5.12. *For all $k \in (0,1)$, there is a constant C_k such that for all positive locally integrable functions f we have if $s + t > -1$:*

$$O_{s,t} f \leq C_k O_{s,t}(R_k^s f),$$

and more generally if $s + t > -1 - N$:

$$O'_{s,t} f \leq C_k O'_{s,t}(R_k^s f).$$

One can find the following lemma in [104] but for $b = Q$.

Lemma 7.5.13. *For all $k \in (0, \frac{1}{2})$, there are two constants C and $k' < 1$ depending only on k, b, Q, N such that for all $f, g \in L^1(d\mu_b), f \geq 0, g \geq 0$:*

$$\int_{\mathbb{B}} f(z)[R_k^b g(z)] d\mu_Q(z) \leq C \int_{\mathbb{B}} g(z)[R_{k'}^{b,Q} f(z)] d\mu_b(z),$$

where

$$R_{k'}^{b,Q}f(z) = \frac{1}{\mu_b(B_{k'}(z))}\int_{B_{k'}(z)} f(\zeta)d\mu_Q(\zeta). \tag{7.5.2.10}$$

Proof. By Lemma 7.5.9, $\chi_{B_k(z)}(w) \leq \chi_{B_{k'}(w)}(z)$, where $k' = \frac{k}{1-k}$. Because of (7.5.2.9) there is a constant C such that

$$\frac{1}{\mu_b(B_k(z))}\chi_{B_k(z)}(w) \leq \frac{C}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z).$$

We want to form the quantity $f(z)[R_k^b g(z)]$ on the left while controlling it on the right in order to use Fubini's theorem to bring out the quantity $g(z)[R_{k'}^{b,Q}f(z)]$. Then, for $w \in B_k(z)$

$$\frac{1}{\mu_b(B_k(z))}\chi_{B_k(z)}(w)g(w) \leq \frac{C}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)g(w).$$

We form $R_k^b g(z)$ on the left

$$\int_{B_k(z)} \frac{1}{\mu_b(B_k(z))}\chi_{B_k(z)}(w)g(w)d\mu_b(w) \leq \int_{B_k(z)} \frac{C}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)g(w)d\mu_b(w)$$

by a multiplication by $f(z)$ we have

$$f(z)R_k^b g(z) \leq Cf(z)\int_{B_k(z)} \frac{1}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)g(w)d\mu_b(w).$$

After integration, we obtain

$$\int_{\mathbb{B}} f(z)R_k^b g(z)d\mu_Q(z) \leq C\int_{\mathbb{B}}\left[\int_{B_k(z)} \frac{1}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)g(w)f(z)d\mu_b(w)\right] d\mu_Q($$

Recall that ($z \in \mathbb{B}$ and $w \in B_k(z)$) $\Longrightarrow$ ($z \in B_{k'}(w)$ and $w \in \mathbb{B}$), hence using Fubini's theorem

$$\int_{\mathbb{B}} f(z)R_k^b g(z)d\mu_Q(z) \leq C\int_{\mathbb{B}} g(w)\left[\int_{B_{k'}(w)} \frac{1}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)f(z)d\mu_Q(z)\right] d\mu_b$$

hence

$$\int_{\mathbb{B}} f(z)R_k^b g(z)d\mu_Q(z) \leq C\int_{\mathbb{B}} g(w)\left[\frac{1}{\mu_b(B_{k'}(w))}\int_{B_{k'}(w)} f(z)d\mu_Q(z)\right] d\mu_b(w)$$

then

$$\int_{\mathbb{B}} f(z)[R_k^b g(z)]d\mu_Q(z) \leq C \int_{\mathbb{B}} g(z)[R_{k'}^{b,Q} f(z)]d\mu_b(z).$$

$\square$

In Lemma 7.5.13 replacing b by s and Q by β we have the following result.

Lemma 7.5.14. *For all $k \in (0, \frac{1}{2})$, there are two constants C and $k' < 1$ depending only on k, s, β, N such that for all $f, g \in L^1(d\mu_s), f \geq 0, g \geq 0$:*

$$\int_{\mathbb{B}} f(z)[R_k^s g(z)]d\mu_\beta(z) \leq C \int_{\mathbb{B}} g(z)[R_{k'}^{s,\beta} f(z)]d\mu_s(z),$$

where

$$R_{k'}^{s,\beta} f(z) = \frac{1}{\mu_s(B_{k'}(z))} \int_{B_{k'}(z)} f(\zeta)d\mu_\beta(\zeta).$$

The following result will be used in the proof of Theorem 7.1.16.

Lemma 7.5.15. *Let $k \in (0, 1)$. There are two constants c, C depending only on a, b, N, k such that for all positive locally integrable functions g if $a > -1$:*

$$c\, m_{a,b} g \leq R_k^b(m_{a,b} g) \leq C m_{a,b} g,$$

and more generally if $a > -1 - N$:

$$c\, m'_{a,b} g \leq R_k^b(m'_{a,b} g) \leq C m'_{a,b} g.$$

Proof. It is sufficient to show that there are two constants $0 < c < C$ such that $\forall w \in B_k(z)$

$$c\, m_{a,b} g(z) \leq m_{a,b} g(w) \leq C m_{a,b} g(z).$$

We are going to show the two inequalities. More precisely we are going to show that there are two constants $0 < c < C$ such that, for each pseudo-ball B

containing z and touching the boundary, there is a pseudo-ball B' containing w and touching the boundary so that

$$\frac{c}{\mu_a(B)} \int_B |g(\zeta)| d\mu_b(\zeta) \leq \frac{1}{\mu_a(B')} \int_{B'} |g(\zeta)| d\mu_b(\zeta)$$

and show for each pseudo-ball B containing z touching the boundary, there is a pseudo-ball B' containing w touching the boundary so that

$$\frac{1}{\mu_a(B)} \int_B |g(\zeta)| d\mu_b(\zeta) \leq \frac{C}{\mu_a(B')} \int_{B'} |g(\zeta)| d\mu_b(\zeta).$$

In each case, by Lemma 7.5.10, it is sufficient if $B = B(x, R)$ to take $B' = B(x, K(1 + 2kK)R)$. For the result with $m'_{a,b}$ it is sufficient to notice that B and B' have equivalent radii. $\square$

In the same way as Lemma 7.5.15, since $O_{s,t}f(z) := (1 - |z|^2)^t m_{s+t,s} f(z)$ and $O'_{s,t}f(z) := (1 - |z|^2)^t m'_{s+t,s} f(z)$, we get:

Lemma 7.5.16. *Let $k \in (0, 1)$. There are two constants c, C depending only on s, t, N, k such that for all locally integrable function g if $s + t > -1$:*

$$c\, O_{s,t}g \leq R^b_k(O_{s,t}g) \leq C O_{s,t}g,$$

and more generally if $s + t > -1 - N$:

$$c\, O'_{s,t}g \leq R^b_k(O'_{s,t}g) \leq C O'_{s,t}g.$$

Now we give a useful characterization of elements in $(B^{a,b,q,Q}_p)$ see Definition 7.1.8.

Lemma 7.5.17. *For $a > -1$, $\omega \in (B^{a,b,q,Q}_p)$ ($b > -1$) if and only if there is a constant $C_{a,b,p,q,Q} > 0$ such that*

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)} \int_B f(z) d\mu_b(z) \right)^p \omega(B) \leq C_{a,b,p,q,Q} \int_B f^p(z) \omega(z) d\mu_q(z).$$

More generally if $a > -1-N$, $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if and only if there is a constant $C_{a,b,p,q,Q} > 0$ such that

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R^{2(N+1+a)}}\int_B f(z)d\mu_b(z)\right)^p \omega(B) \leq C_{a,b,p,q,Q}\int_B f^p(z)\omega(z)d\mu_q(z),$$

for all positive $f \in L^p(\omega d\mu_q)$ and all pseudo balls B of radius R such that $B \cap \partial\mathbb{B} \neq \varnothing$. Here $\omega(B) = \displaystyle\int_B \omega(z)d\mu_Q(z)$.

Proof. Assume $\omega \in (B_p^{a,b,q,Q})$. Let $0 \leq f \in L^p(\omega d\mu_q)$ and B be a pseudo ball such that $B \cap \partial\mathbb{B} \neq \varnothing$. Then

$$\begin{aligned}
\left(\int_B f(z)d\mu_b(z)\right)^p &= \left(\int_{\mathbb{B}} f(z)(1-|z|^2)^{b-q}d\mu_q(z)\right)^p \\
&= \left(\int_B f(z)(\omega(z))^{\frac{-1}{p}}(\omega(z))^{\frac{1}{p}}(1-|z|^2)^{b-q}d\mu_q(z)\right)^p \\
&\leq \left(\int_B f^p(z)\omega(z)d\mu_q(z)\right)\Big(\int_B ((\omega(z))^{\frac{-1}{p}})^{p'}(1-|z|^2)^{p'(b-q)}d\mu_q(\\
&= \left(\int_B f^p(z)\omega(z)d\mu_q(z)\right)\left(\int_B \omega^{\frac{-1}{p-1}}(z)d\mu_{q+p'(b-q)}(z)\right)^{p-1}.
\end{aligned}$$

Hence

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B f(z)d\mu_b(z)\right)^p \omega(B) \leq$$

$$\left(\int_B f^p(z)\omega(z)d\mu_q(z)\right)\left[\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\omega(B)\right]\Big(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B \omega^{\frac{-1}{p-1}}(z)d\mu_{q+p'(b-q)}(z$$

and because $\omega \in B_p^{a,b,q,Q}$, there is a constant $C_{a,b,p,q,Q} > 0$ such that

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B f(z)d\mu_b(z)\right)^p \omega(B) \leq C_{a,b,p,q,Q}\int_B f^p(z)\omega(z)d\mu_q(z).$$

For the general case it is sufficient to replace $\mu_a(B)$ with R^{N+1+a}.

If we assume that there is a constant $C_{a,b,p,q,Q} > 0$ such that

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R^{2(N+1+a)}}\int_B f(z)d\mu_b(z)\right)^p \omega(B) \leq C_{a,b,p,q,Q}\int_B f^p(z)\omega(z)d\mu_q(z),$$

for all positive $f \in L^p(\omega d\mu_q)$ and all pseudo balls B of radius R such that $B \cap \partial\mathbb{B} \neq \varnothing$, it is sufficient to take $f(z) = (1-|z|^2)^{(p'-1)(b-q)}\omega^{\frac{-1}{p-1}}(z)\chi_B(z)$ to get $\omega \in (B_p^{a,b,q,Q})$. $\square$

Remark 7.5.18. The result remains true even if B almost touches the edge.

In the same way, for $(D_p^{s,t,q,Q})$ (see Definition 7.1.13), we have the following lemma.

Lemma 7.5.19. *For $Q \geq q$ and $s+t > -1$, $\omega \in (D_p^{s,t,q,Q})$ $(s > -1)$ if and only if then there is a constant $C_{s,t,p,q,Q} > 0$ such that*

$$\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)}\int_B f(z)d\mu_s(z)\right)^p \int_B \omega(z)d\mu_{Q+pt}(z) \leq C_{s,t,p,q,Q}\int_B f^p(z)\omega(z)d\mu_q(z$$

for all positive $f \in L^p(\omega d\mu_q)$ and all pseudo balls B such that $B \cap \partial\mathbb{B} \neq \varnothing$.

For $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -1-N$, $\omega \in (D_p^{s,t,q,Q})$ $(s > -1)$ if and only if then there is a constant $C_{s,t,p,q,Q} > 0$ such that

$$\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)}\int_B f(z)d\mu_s(z)\right)^p \int_B \omega(z)d\mu_{Q+pt}(z) \leq C_{s,t,p,q,Q}\int_B f^p(z)\omega(z)d\mu_q(z$$

for all positive $f \in L^p(\omega d\mu_q)$ and all pseudo balls B such that $B \cap \partial\mathbb{B} \neq \varnothing$.

Remark 7.5.20. The result remains true even if B almost touches the edge.

Corollary 7.5.21. *For $C_1 > 1$, if $\omega \in D_p^{s,t,q,Q}$ then there exists a constant $C_2 > 0$ such that for any pseudo-ball $B := B(y,r)$ which touches or almost touches the edge, we have:*

$$\int_{B(y,C_1 r)} \omega(\zeta)d\mu_{Q+pt}(\zeta) \leq C_2 \int_{B(y,r)} \omega(\zeta)d\mu_{Q+pt}(\zeta).$$

Proposition 7.5.22. *Let X be an homogeneous space. Let w be a weight in X. For $a \leq b$, assume that there exists a constant $C_2 > 0$ such that*

$$\left(\int_B [M_\gamma(\chi_B u^{1-p'})(x)]^r v(x) d\nu(x)\right)^{\frac{1}{r}} \leq C_2 \left(\int_B u^{1-p'}(x) d\nu(x)\right)^{\frac{1}{p}} \quad (7.5.2.11)$$

for any pseudo-ball $B \subset X$, where $v(z) = R_{k'}^{b,Q}\omega(z)$, $u(z) = R_{k'}^{b,Q}\omega(z)(1 - |z|^2)^{2p(b-a)+Q-q}$, $d\nu = d\mu_b$ $p = r$ and $\gamma = 1 - \frac{N+1+a}{N+1+b}$. If $\omega \in (B_p^{a,b,q,Q})$, there is a constant $C_{a,b,p,q,Q} > 0$ such that $\forall f \in L^p(\omega d\mu_q)$,

$$\int_{\mathbb{B}} (m_{a,b} f(z))^p \, \omega(z) d\mu_Q(z) \leq \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z).$$

Proof. Let set

$$III = \int_{\mathbb{B}} (m_{a,b} f(z))^p \, \omega(z) d\mu_Q(z).$$

Using in this order Lemma 7.5.11, Lemma 7.5.15, Hölder's inequality and Lemma 7.5.13 we have,

$$\begin{aligned}
III &\leq C_k^p \int_{\mathbb{B}} (m_{a,b} R_k^b f(z))^p \omega(z) d\mu_Q(z) \\
&\leq C_k^p A^p \int_{\mathbb{B}} \left[R_k^b (m_{a,b} R_k^b f(z))\right]^p \omega(z) d\mu_Q(z) \\
&\leq C_k^p A^p \int_{\mathbb{B}} R_k^b \left[(m_{a,b} R_k^b f(z))^p\right] \omega(z) d\mu_Q(z) \\
&\leq C_k^p A^p C \int_{\mathbb{B}} (m_{a,b} R_k^b f(z))^p R_{k'}^{b,Q} \omega(z) d\mu_b(z).
\end{aligned}$$

For $a \leq b$, assume that there exists a constant $C_2 > 0$ such that

$$\left(\int_B [M_\gamma(\chi_B u^{1-p'})(x)]^r v(x) d\nu(x)\right)^{\frac{1}{r}} \leq C_2 \left(\int_B u^{1-p'}(x) d\nu(x)\right)^{\frac{1}{p}}$$

for any ball $B \subset X$, where $v(z) = R_{k'}^{b,Q}\omega(z)$, $u(z) = R_{k'}^{b,Q}\omega(z)(1 - |z|^2)^c$, $d\nu =$

$d\mu_b$ $p = r$ and $\gamma = 1 - \frac{N+1+a}{N+1+b}$, with c is to be determined. Then we have

$$\begin{aligned}
\int_{\mathbb{B}} (m_{a,b} f(z))^p \omega(z) d\mu_Q(z) &\leq C_k^p A^p C \int_{\mathbb{B}} (m_{a,b} R_k^b f(z))^p R_{k'}^{b,Q} \omega(z) d\mu_b(z) \\
&\leq C_k^p A^p C \int_{\mathbb{B}} (M_{a,b} R_k^b f(z))^p R_{k'}^{b,Q} \omega(z) d\mu_b(z) \\
&\lesssim C_k^p A^p C \int_{\mathbb{B}} (M_\gamma R_k^b f(z))^p R_{k'}^{b,Q} \omega(z) d\mu_b(z) \\
&\leq C_k^p A^p C' \int_{\mathbb{B}} (R_k^b f(z))^p (1-|z|^2)^c R_{k'}^{b,Q} \omega(z) d\mu_b(z),
\end{aligned}$$

where for the last inequality we used Theorem 7.2.9. Now let us control $(R_k^b f(z))^p R$

We have

$$\begin{aligned}
(R_k^b f(z))^p R_{k'}^{b,Q} \omega(z) &= \left(\frac{1}{\mu_b(B_k(z))} \int_{B_k(z)} f(\zeta) d\mu_b(\zeta) \right)^p \left(\frac{1}{\mu_b(B_{k'}(z))} \int_{B_{k'}(z)} \omega(\zeta) d \right. \\
&\leq \left(\frac{1}{\mu_b(B_k(z))} \int_{B_{k'}(z)} f(\zeta) d\mu_b(\zeta) \right)^p \left(\frac{1}{\mu_b(B_{k'}(z))} \int_{B_{k'}(z)} \omega(\zeta) \right. \\
&\lesssim \left(\frac{1}{\mu_b(B_{k'}(z))} \int_{B_{k'}(z)} f(\zeta) d\mu_b(\zeta) \right)^p \left(\frac{1}{\mu_b(B_{k'}(z))} \int_{B_{k'}(z)} \omega(\zeta) \right.
\end{aligned}$$

where the second inequality is because $k' > k$, the third one is because $\mu_b(B_{k'}(z)) \backsimeq$

$\mu_b(B_k(z))$. Then

$$(R_k^b f(z))^p R_{k'}^{b,Q} \omega(z) \lesssim$$

$$\frac{\mu_a^{2p}(B_{k'}(z))}{\mu_b^{p+1}(B_{k'}(z)) \mu_{b+\frac{Q-q}{p}}^p (B_{k'}(z))} \left(\frac{\mu_{b+\frac{Q-q}{p}}(B_{k'}(z))}{\mu_a^2(B_{k'}(z))} \int_{B_{k'}(z)} f(\zeta) d\mu_b(\zeta) \right)^p \left(\int_{B_{k'}(z)} \omega(\right.$$

so that

$$(R_k^b f(z))^p R_{k'}^{b,Q} \omega(z) \lesssim C_{a,b,p,q,Q} \frac{\mu_a^{2p}(B_{k'}(z))}{\mu_b^{p+1}(B_{k'}(z)) \mu_{b+\frac{Q-q}{p}}^p (B_{k'}(z))} \int_{B_{k'}(z)} f^p(\zeta) \omega(\zeta) d\mu_q$$

because of Lemma 7.5.17 since it is possible to dilate the pseudo-balls B_k so that they touch the boundary and the fact that the measures $d\mu_q$ and $d\mu_{q+p'(b-a)}$ are homogeneous. By Lemma 7.2.3 we have

$$\frac{\mu_a^{2p}(B_{k'}(z))}{\mu_b^{p+1}(B_{k'}(z)) \mu_{b+\frac{Q-q}{p}}^p (B_{k'}(z))} \backsimeq (1-|z|^2)^{2pa-2pb-b-(Q-q)-(N+1)}.$$

Recall that we already have

$$\int_{\mathbb{B}}(m_{a,b}f(z))^p\omega(z)d\mu_Q(z) \leq C_k^p A^p C' \int_{\mathbb{B}}(1-|z|^2)^{c+(b-a)}(R_k^b f(z))^p R_{k'}^{b,Q}\omega(z)d\mu_a(z).$$

Let us set

$$IV = \int_{\mathbb{B}}(1-|z|^2)^{c+(b-a)}(R_k^b f(z))^p R_{k'}^{b,Q}\omega(z)d\mu_a(z).$$

Hence, using the previous control of $(R_k^b f(z))^p R_{k'}^{b,Q}\omega(z)$ and Fubini's theorem, we have

$$\begin{aligned}
IV &\lesssim \int_{\mathbb{B}}\left(\int_{B_{k'}(z)} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\right)(1-|z|^2)^{c+(b-a)+2pa-2pb-b-Q+q-1-N}d\mu_a(z)\\
&\lesssim \int_{\mathbb{B}}\left(\int_{\mathbb{B}}\chi_{B_{k'}(z)}(\zeta)(1-|z|^2)^{c+(1-2p)(b-a)-b-Q+q-1-N}d\mu_a(z)\right)f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\\
&\lesssim \int_{\mathbb{B}}\left(\int_{\mathbb{B}}\chi_{B_{k''}(\zeta)}(z)(1-|z|^2)^{c+(1-2p)(b-a)-b-Q+q-1-N}d\mu_a(z)\right)f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\\
&\lesssim \int_{\mathbb{B}}\left(\int_{\mathbb{B}}\chi_{B_{k''}(\zeta)}(z)(1-|z|^2)^{c-2p(b-a)-Q+q-1-N}d\mu(z)\right)f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\\
&\lesssim \int_{\mathbb{B}}(1-|\zeta|^2)^{c-2p(b-a)-(Q-q)}f^p(\zeta)\omega(\zeta)d\mu_q(\zeta).
\end{aligned}$$

The proof is complete if we take

$$c = 2p(b-a)+(Q-q).$$

□

Remark 7.5.23. The result in Proposition 7.5.22 says that if we assume that the Sawyer type condition (7.5.2.11) holds, then the necessary condition $w \in (B_p^{a,b,q,Q})$ for the boundedness of $T_{a,b}$ from $L^p(wd\mu_q)$ to $L^p(wd\mu_Q)$ is also sufficient by the good lambda inequality in Theorem 7.6.3. Since we do not know if $w \in (B_p^{a,b,q,Q})$ implies the Sawyer type condition, we will provide in the sequel a testable sufficient condition for the boundedness of $T_{a,b}$ in this situation.

Lemma 7.5.24. *For $s+t>-1$, or for $-N-1<s+t<-1$ with $s+t+\frac{Q-q}{p}>-1$, if $\omega\in D_p^{s,t,q,Q}$, we have $\sigma(z)=R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}}\in(A_{p,s+t+\frac{Q-q}{p}})$.*

Proof. We have $\sigma(z)=R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}}\in(A_{p,s+t+\frac{Q-q}{p}})$ if

$$\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)}\int_B\sigma(z)d\mu_{s+t+\frac{Q-q}{p}}(z)\right)\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)}\int_B\sigma^{\frac{-1}{p-1}}(z)d\mu_{s+t+\frac{Q-q}{p}}(z)\right)$$

Note that $\sigma(z)\simeq R_{k'}^{s+t+\frac{Q-q}{p},Q+pt}\omega(z)$ since $d\mu_b(B_{k'}(z))\simeq(1-|z|^2)^{n+1+b}$. We consider two cases.

First case: $B:=B(y,r)$ with $r\ll 1-|y|$.

In this case, because of Corollary 7.5.21, there are two constants 0¡c¡C such that

$$cR_{k'}^{s+t+\frac{Q-q}{p},Q+pt}\omega(y)<R_{k'}^{s+t+\frac{Q-q}{p},Q+pt}\omega(x)<CR_{k'}^{s+t+\frac{Q-q}{p},Q+pt}\omega(y)$$

for all $x\in B$. Then setting

$$V=\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)}\int_B\sigma(z)d\mu_{s+t+\frac{Q-q}{p}}(z)\right)\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)}\int_B\sigma^{\frac{-1}{p-1}}(z)d\mu_{s+t+\frac{Q-q}{p}}\right.$$

we have

$$V\simeq\frac{\mu_{s+t+\frac{Q-q}{p}}(B)}{\mu_{s+t+\frac{Q-q}{p}}(B)}\left(\frac{\mu_{s+t+\frac{Q-q}{p}}(B)}{\mu_{s+t+\frac{Q-q}{p}}(B)}\right)^{p-1}=1.$$

Second case: $B:=B(y,r)$ touches the edge.

Recall that our measures are homogeneous, and recall that if $z\in B$ and $x\in B_{k'}(z)$, then $z\in B_{k''}(x)$ and $x\in B':=B(y,2k'Kr+Kr)$. Let

$$VI=\int_B\left(\frac{1}{\mu_s(B_{k'}(z))}\int_{B_{k'}(z)}\omega(x)d\mu_{Q+pt}(x)\right)(1-|z|^2)^{-t-\frac{Q-q}{p}}d\mu_{s+t+\frac{Q-q}{p}}(z).$$

Then by Fubini's theorem we have,

$$\begin{aligned}
VI &= \int_B \left(\frac{1}{\mu_s(B_{k'}(z))} \int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right) d\mu_s(z) \\
&= \int_B \int_{B_{k'}(z)} \frac{1}{\mu_s(B_{k'}(z))} \omega(x) d\mu_s(z) d\mu_{Q+pt}(x) \\
&= \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{1}{\mu_s(B_{k'}(z))} \chi_B(z) \chi_{B_{k'}(z)}(x) \omega(x) d\mu_s(z) d\mu_{Q+pt}(x) \\
&\lesssim \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{1}{\mu_s(B_{k''}(x))} \chi_{B'}(x) \chi_{B_{k''}(x)}(z) \omega(x) d\mu_s(z) d\mu_{Q+pt}(x) \\
&\lesssim \int_{B'} \omega(x) d\mu_{Q+pt}(x).
\end{aligned}$$

So we have

$$VI = \int_B R_{k'}^{s,Q+pt} \omega(z) (1-|z|^2)^{-t-\frac{Q-q}{p}} d\mu_{s+t+\frac{Q-p}{p}}(z) \lesssim \int_{B'} \omega(x) d\mu_{Q+pt}(x). \tag{7.5.2.12}$$

Let us now control $(R_{k'}^{s,Q+pt} \omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}})^{1-p'}$. We have

$$\begin{aligned}
(R_{k'}^{s,Q+pt} \omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}})^{1-p'} &= \left(\frac{1}{\mu_s(B_{k'}(z))} (1-|z|^2)^{-t-\frac{Q-q}{p}} \int_{B_{k'}(z)} \omega(x) d\mu \right. \\
&\simeq \left[\left(\int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left(\int_{B_{k'}(z)} d\mu_{s+} \right. \right.
\end{aligned}$$

Setting

$$VII = \left[\left(\int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left(\int_{B_{k'}(z)} d\mu_{s+t+\frac{Q-q}{p}}(x) \right) \right]^{p'-1},$$

we have, by Hölder's inequality

$$\begin{aligned}
VII &= \left[\left(\int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left(\int_{B_{k'}(z)} \omega^{\frac{1}{p}}(x) \omega^{\frac{-1}{p}}(x) (1-|x|^2)^{s-\frac{q}{p}+\frac{Q+pt}{p}} d\mu(\right. \right. \\
&\leq \left[\left(\int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{\frac{1}{p}-1} \left(\int_{B_{k'}(z)} \omega^{\frac{-p'}{p}}(x) d\mu_{q+p'(s-q)}(x) \right)^{\frac{1}{p'}} \right]^{p'-1} \\
&\leq \left[\left(\int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left(\int_{B_{k'}(z)} \omega^{\frac{-p'}{p}}(x) d\mu_{q+p'(s-q)}(x) \right) \right]^{\frac{1}{p}}.
\end{aligned}$$

Let us set

$$VIII = \int_B \left(R^{s,Q+pt}_{k'}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}}\right)^{\frac{-1}{p-1}} d\mu_{s+t+\frac{Q-q}{p}}(z),$$

then we have

$$VIII \leq \int_B \left[\left(\int_{B_{k'}(z)} \omega(x)d\mu_{Q+pt}(x)\right)^{-1} \left(\int_{B_{k'}(z)} \omega^{\frac{-p'}{p}}(x)d\mu_{q+p'(s-q)}(x)\right) \omega(z\right.$$

$$\omega^{\frac{-1}{p}}(z)(1-|z|^2)^{s-\frac{q}{p}}d\mu(z)$$

so that by Hölder's inequality and Fubini's theorem, we have

$$VIII \lesssim \left(\int_B \omega^{\frac{-p'}{p}}(z)d\mu_{q+p'(s-q)}(z)\right)^{\frac{1}{p'}}$$

$$\left(\int_{B'}\left[\left(\int_{B_{k''}(x)} \omega(\zeta)d\mu_{Q+pt}(\zeta)\right)^{-1} \left(\int_{B_{k''}(x)} \omega(z)d\mu_{Q+pt}(z)\right)\right] \omega^{\frac{-p'}{p}}(x)d\mu_{q+p'(s-q}\right.$$

Finally, we obtain

$$VIII \lesssim \int_{B'} \omega^{\frac{-p'}{p}}(z)d\mu_{q+p'(s-q)}(z). \tag{7.5.2.13}$$

Where on the last but one inequality we used Fubini's theorem (as in the control of VI) and the fact that for $x \in B_{k'}(z)$ we have $z \in B_{k''}(x)$ and

$$\int_{B_{k''}(x)} \omega(\zeta)d\mu_{Q+pt}(\zeta) \lesssim \int_{B_{k'}(z)} \omega(\zeta)d\mu_{Q+pt}(\zeta).$$

This is a variant of Corollary 7.5.21 or simply the application of Lemma 7.5.19 for $f(\zeta) = (1-|\zeta|^2)^{t+\frac{Q-q}{p}} 1_{B_{k'}(z)}(\zeta)$ with $B := B(z, 2Kk'(1+k")(1-|z|)) \supseteqq B_{k''}(x)$ (Lemma 7.5.10) and the fact that our measures are homogeneous. Since $\omega \in D^{s,t,q,Q}_p$, we use (7.5.2.12) and (7.5.2.13) to conclude that $\sigma \in (A_{p,s+t+\frac{Q-q}{p}})$.

$\square$

Theorem 7.5.25. *In the case both $Q \geq q$ and $s+t > -1$ hold, and in the case both $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$ hold, if $\omega \in D^{s,t,q,Q}_p$, there is* *a constant* $C_{s,t,p,q,Q} > 0$ *such that* $\forall f \in L^p(\omega d\mu_q)$,

$$\int_{\mathbb{B}} (O_{s,t} f(z))^p \omega(z) d\mu_Q(z) \leq C_{s,t,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z).$$

Proof. Using in this order Lemma 7.5.12, Lemma 7.5.16, Hölder's inequality and Lemma 7.5.14 we have

$$\begin{aligned}
\int_{\mathbb{B}} (O_{s,t} f(z))^p \omega(z) d\mu_Q(z) &= \int_{\mathbb{B}} (m_{s+t,s} f(z))^p \omega(z) d\mu_{Q+pt}(z) \\
&\leq C_k^p \int_{\mathbb{B}} (m_{s+t,s} R_k^s f(z))^p \omega(z) d\mu_{Q+pt}(z) \\
&\leq C_k^p A^p \int_{\mathbb{B}} [R_k^s (m_{s+t,s} R_k^s f(z))]^p \omega(z) d\mu_{Q+pt}(z) \\
&\leq C_k^p A^p \int_{\mathbb{B}} R_k^s [(m_{s+t,s} R_k^s f(z))^p] \omega(z) d\mu_{Q+pt}(z) \\
&\leq C_k^p A^p C \int_{\mathbb{B}} (m_{s+t,s} R_k^s f(z))^p R_{k'}^{s,Q+pt} \omega(z) d\mu_s(z).
\end{aligned}$$

By Lemma 7.5.24 $R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}} \in A_{p,s+t+\frac{Q-q}{p}}$. Using natural domination between our maximal operator defined by equation 7.5.2.1 and equation 7.5.2.3, and using Theorem 7.2.13, we have

$$\begin{aligned}
\int_{\mathbb{B}} (O_{s,t} f(z))^p \omega(z) d\mu_Q(z) &\leq C_k^p A^p C \int_{\mathbb{B}} (m_{s+t,s} R_k^s f(z))^p R_{k'}^{s,Q+pt} \omega(z) d\mu_s(z) \\
&\leq C_k^p A^p C \int_{\mathbb{B}} (M_{s+t,s} R_k^s f(z))^p R_{k'}^{s,Q+pt} \omega(z) d\mu_s(z) \\
&= C_k^p A^p C \int_{\mathbb{B}} (M_{s+t,s+t} [(1-|z|^2)^{-t} R_k^s f(z)])^p R_{k'}^{s,Q+pt} \omega(z
\end{aligned}$$

Because in each case we have $\frac{Q-q}{p} \geq 0$, we have that

$$\begin{aligned}
\int_{\mathbb{B}} (O_{s,t} f(z))^p \omega(z) d\mu_Q(z) &\lesssim \int_{\mathbb{B}} (M_{s+t+\frac{Q-q}{p},s+t} [(1-|z|^2)^{-t} R_k^s f(z)])^p R_{k'}^{s,Q+pt} \omega(z) d\mu \\
&\lesssim \int_{\mathbb{B}} (M_{s+t+\frac{Q-q}{p},s+t+\frac{Q-q}{p}} [(1-|z|^2)^{-t-\frac{Q-q}{p}} R_k^s f(z)])^p R_{k'}^{s,Q} \\
&\lesssim \int_{\mathbb{B}} (1-|z|^2)^{-pt-(Q-q)} (R_k^s f(z))^p R_{k'}^{s,Q+pt} \omega(z) d\mu_s(z).
\end{aligned} \tag{7.5.2.14}$$

Now let us control $IX = (R_k^s f(z))^p R_{k'}^{s,Q+pt}\omega(z)$. We have

$$
\begin{aligned}
IX &= \left(\frac{1}{\mu_s(B_k(z))}\int_{B_k(z)} f(\zeta)d\mu_s(\zeta)\right)^p \left(\frac{1}{\mu_s(B_{k'}(z))}\int_{B_{k'}(z)} \omega(\zeta)d\mu_{Q+pt}(\zeta)\right)\\
&\lesssim \left(\frac{1}{\mu_s(B_{k'}(z))}\int_{B_{k'}(z)} f(\zeta)d\mu_s(\zeta)\right)^p \left(\frac{1}{\mu_s(B_{k'}(z))}\int_{B_{k'}(z)} \omega(\zeta)d\mu_{Q+pt}(\zeta)\right)\\
&\lesssim \frac{\mu^p_{s+t+\frac{Q-q}{p}}(B_{k'}(z))}{\mu_s^{p+1}(B_{k'}(z))}\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B_{k'}(z))}\int_{B_{k'}(z)} f(\zeta)d\mu_s(\zeta)\right)^p \left(\int_{B_{k'}(z)} \omega(\zeta)d\right.\\
&\lesssim C_{s,t,p,q,Q}\frac{\mu^p_{s+t+\frac{Q-q}{p}}(B_{k'}(z))}{\mu_s^{p+1}(B_{k'}(z))}\int_{B_{k'}(z)} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\\
&\lesssim C_{s,t,p,q}(1-|z|^2)^{pt+(Q-q)-s-N-1}\int_{B_{k'}(z)} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta),
\end{aligned}
$$

where for the last but one inequality we used Lemma 7.5.19. Hence using (7.5.2.14) and Fubini's theorem we have

$$
\begin{aligned}
\int_{\mathbb{B}}(O_{s,t}f(z))^p\omega(z)d\mu_Q(z) &\leq \int_{\mathbb{B}}(1-|z|^2)^{-pt-(Q-q)}(R_k^s f(z))^p R_{k'}^{s,Q+pt}\omega(z)d\mu_s(z)\\
&\lesssim \int_{\mathbb{B}}\left(\int_{B_{k'}(z)} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\right)(1-|z|^2)^{-1-N}d\mu(z)\\
&\lesssim \int_{\mathbb{B}}\left(\int_{\mathbb{B}}\chi_{B_{k''}(\zeta)}(z)(1-|z|^2)^{-1-N}d\mu(z)\right)f^p(\zeta)\omega(\zeta)d\mu_q\\
&\lesssim \int_{\mathbb{B}} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta).
\end{aligned}
$$

Here we used Lemma 7.5.9 with $k'' := (k')'$. The proof is complete. $\square$

7.6 Good lambda inequality and sufficient conditions

In this section we will establish the good lambda inequality that allow us to provide sufficient conditions for the boundedness of our operators. We first need some preliminary results.

Proposition 7.6.1. *Let $\beta > 0$ and $s+t > -1-N$ there is a constant $A > 0$ such that for all $\xi \in \mathbb{D}$*

$R > 1-|z_0|$ and a positive locally integrable function f and for all $z \in B(z_0, R)$ if $s+t > -1$:

$$R^{\beta} \int_{d(z_0,\xi)\geq R} \frac{f(\xi)}{d(z_0,\xi)^{N+1+s+t+\beta}} \mathrm{d}\mu_s(\xi) \leq A m_{s+t,s} f(z).$$

More generally if $-1-N < s+t$ then:

$$R^{\beta} \int_{d(z_0,\xi)\geq R} \frac{f(\xi)}{d(z_0,\xi)^{N+1+s+t+\beta}} \mathrm{d}\mu_s(\xi) \leq A m'_{s+t,s} f(z).$$

Proof. Recall that if $s+t > -1$, by Lemma 7.2.3, there is a constant $a > 0$ such that for all $k \in \mathbb{N}$, we have $\mu_{s+t}(B(z_0, 2^{k+1}R)) \leq a(2^{k+1}R)^{N+1+s+t}$, so that setting

$$X = R^{\beta} \int_{d(z_0,\xi)\geq R} \frac{f(\xi)}{d(z_0,\xi)^{N+1+s+t+\beta}} \mathrm{d}\mu_s(\xi)$$

we have

$$\begin{aligned} X &\leq \sum_{k=0}^{+\infty} \frac{1}{2^{k(N+1+s+t+\beta)} R^{N+1+s+t}} \int_{d(z_0,\xi)<2^{k+1}R} f(\xi)\mathrm{d}\mu_s(\xi) \\ &\leq a2^{N+1+s+t} \sum_{k=0}^{+\infty} 2^{-k\beta} \frac{1}{\mu_{s+t}(B(z_0, 2^{k+1}R))} \int_{B(z_0,2^{n+1}R)} f(\xi)\mathrm{d}\mu_s(\xi) \\ &\leq a2^{N+1+s+t} m_{s+t,s} f(z) \sum_{n=0}^{+\infty} 2^{-k\beta} = \frac{a2^{N+1+s+t}}{1-2^{-\beta}} m_{s+t,s} f(z). \end{aligned}$$

We can take $A = \dfrac{a2^{N+1+s+t}}{1-2^{-\beta}}$ to conclude. □

Proposition 7.6.2. *Let $\omega \in (D_p^{s,t,q,Q})$, we set again $\sigma(z) = R_k^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}}$ with $k \in (0, 1/2)$. Set $B = B(z', r)$ with $1-|z'| < cr$ and $L = \left\{z \in B : 1-|z| < C'_0 \gamma^{\frac{1}{N+1+s+t}} r\right\}$ where $C'_0 > 0$, $0 < \gamma < 1$, $r > 0$ and $c > 0$ are constants. Then if we set $L' = \left\{z \in \bar{B} : 1-|z| < 2C'_0 \gamma^{\frac{1}{N+1+s+t}} r\right\}$ and $\bar{B} = B(z', ar)$ with $a = K(C'_0+1)$, there are two constants C_1 et $C_2 > 0$ independent* *of γ such that if $s+t+\frac{Q-q}{p} > -1$ then:*

$$\omega d\mu_{Q+pt}(L) \leq C_1 \sigma d\mu_{s+t+\frac{Q-q}{p}}(L') \quad \text{and} \quad \mu_{s+t+\frac{Q-q}{p}}(L') \leq C_2 \gamma^{\frac{s+t+\frac{Q-q}{p}+1}{N+1+s+t}} \mu_{s+t+}$$

(7.6.0.15)

Proof. Let $k_1 = \frac{k}{1+k}$ then for $z \in L$ and $\xi \in B_{k_1}(z)$ we have $z \in B = B(z', r)$, $1-|z| < C_0' \gamma^{\frac{1}{N+1+s+t}} r$ and $z \in B_k(\xi)$ because $k_1' = k$. Then

$$d(z',\xi) \leq K(d(z',z)+d(z,\xi)) < K[r+k_1(1-|z|)] < K[r+k_1 C_0' \gamma^{\frac{1}{N+1+s+t}} r] < (C_0'+1$$

because $0 < k_1, \gamma < 1$. Then $\xi \in \overline{B} = B(z', ar)$ with $a = K(C_0'+1)$. Moreover,

$$1-|\xi| < 1-|z| + d(z,\xi) < (k_1+1)(1-|z|) < 2C_0' \gamma^{\frac{1}{N+1+s+t}} r$$

because $0 < k_1 < 1$, so that $\xi \in L' = \left\{ z \in \overline{B} : 1-|z| < 2C_0' \gamma^{\frac{1}{N+1+s+t}} r \right\}$. Then we have

$$\chi_L(z) \chi_{B_{k_1}(z)}(\xi) \leq \chi_{L'}(\xi) \chi_{B_k(\xi)}(z).$$

Remember that

$$\mu_{s+t+\frac{Q-q}{p}}(B_k(\xi)) \simeq \mu_{s+t+\frac{Q-q}{p}}(B_{k_1}(z)).$$

Hence,

$$\begin{aligned}
\omega d\mu_{Q+pt}(L) &= \int_L \omega(z) d\mu_{Q+pt}(z) \\
&= \int_L \left(\frac{\omega(z)}{\mu_s(B_{k_1}(z))} \int_{B_{k_1}(z)} \mathrm{d}\mu_s(\xi) \right) d\mu_{Q+pt}(z) \\
&= \int_{\mathbb{B}} \left(\int_{\mathbb{B}} \frac{\chi_L(z)\chi_{B_{k_1}(z)}(\xi)\omega(z)}{\mu_s(B_{k_1}(z))} \mathrm{d}\mu_{Q+pt}(z) \right) \mathrm{d}\mu_s(\xi) \\
&\lesssim \int_{\mathbb{B}} \left(\int_{\mathbb{B}} \frac{\chi_{L'}(\xi)\chi_{B_k(\xi)}(z)\omega(z)}{\mu_s(B_k(\xi))} \mathrm{d}\mu_{Q+pt}(z) \right) \mathrm{d}\mu_s(\xi) \\
&= \int_{\mathbb{B}} \left(\frac{\chi_{L'}(\xi)}{\mu_s(B_k(\xi))} \int_{B_k(\xi)} \omega(z) \mathrm{d}\mu_{Q+pt}(z) \right) (1-|\xi|^2)^{-t-\frac{Q-q}{p}} \mathrm{d}\mu_{s+t+} \\
&= \int_{L'} \left(R_k^{s,Q+pt} w(\xi) \right) (1-|\xi|^2)^{-t-\frac{Q-q}{p}} \mathrm{d}\mu_{s+t+\frac{Q-q}{p}}(\xi) \\
&= \sigma d\mu_{s+t+\frac{Q-q}{p}}(L').
\end{aligned}$$

Let us show the second inequality. Let $z \in \overline{B}$, then, $d(z', z) = ||z'| - |z|| + \left|1 - \frac{\langle z', z\rangle}{|z'||z|}\right| < ar$. Then $||z'| - |z|| < ar$ and $\left|1 - \frac{\langle z', z\rangle}{|z'||z|}\right| < ar$. Moreover, as $1 - |z'| < cr$ then: $1 - |z| = 1 - |z'| + |z'| - |z| < cr + ar$. Then for $z \in L'$. Setting $\beta_1 = 2C_0'\gamma^{\frac{1}{N+1+s+t}} r$ and $\beta_2 = (c+a)r$, we get $1 - |z| < \beta_1$, $1 - |z| < \beta_2$ and $\left|1 - \frac{\langle z', z\rangle}{|z'||z|}\right| < ar$. Then, $L' \subset \{z \in \mathbb{B} : 1 - |z| < \min(\beta_1, \beta_2),\ \left|1 - \frac{\langle z', z\rangle}{|z'||z|}\right| < ar\}$. In spherical coordinates we have, for $s + t + \frac{Q-q}{p} > -1$

$$\begin{aligned}
\mu_{s+t+\frac{Q-q}{p}}(L') &\lesssim (s+t+\frac{Q-q}{p}+1)\int_{1-\rho<\min(\beta_1,\beta_2)} (1-\rho^2)^{s+t+\frac{Q-q}{p}}\rho \mathrm{d}\rho \int_{|1-\frac{\langle z', z\rangle}{|z'||z|}|} \\
&\leq (s+t+\frac{Q-q}{p}+1)\int_{1-\min(\beta_1,\beta_2)<\rho<1} (1-\rho)^{s+t+\frac{Q-q}{p}} \mathrm{d}\rho \int_{|1-\frac{\langle z', z\rangle}{|z'||z|}|} \\
&\lesssim r^N[-(1-\rho)^{s+t+\frac{Q-q}{p}+1}]_{1-\min(\beta_1,\beta_2)}^1 = r^N(\min(\beta_1,\beta_2))^{s+t+\frac{Q-q}{p}} \\
&\leq r^N\beta_1^{s+t+\frac{Q-q}{p}+1} = r^N(2C_0'\gamma^{\frac{1}{N+1+s+t}}r)^{s+t+\frac{Q-q}{p}+1} \backsimeq r^{N+s+t+\frac{Q-}{p}}
\end{aligned}$$

then,

$$\mu_{s+t+\frac{Q-q}{p}}(L') \lesssim r^{N+1+s+t+\frac{Q-q}{p}}\gamma^{\frac{s+t+\frac{Q-q}{p}+1}{N+1+s+t}}. \tag{7.6.0.16}$$

In other hand, as $1 - |z'| < cr$, we have by Lemma 7.2.3

$$\mu_{s+t+\frac{Q-q}{p}}(\bar{B}) \simeq (ar)^{N+1}(\max(1-|z'|, ar))^{s+t+\frac{Q-q}{p}} \lesssim r^{N+1+s+t+\frac{Q-q}{p}}.$$

□

The following result is used to show that $S_{s+t,s}f \in L^p(\omega d\mu_{Q+pt})$ when $m'_{s+t,s}f \in L^p(\omega d\mu_{Q+pt})$.

Theorem 7.6.3 (Good lambda inequality). *Let $\omega \in (D_p^{s,t,q,Q})$ $(1 < p < +\infty)$ in the case both $s + t + \frac{Q-q}{p} > -1$ and $-1 > s + t > -N - 1$ hold, or both $s + t > -1$ and $Q \geq q$ hold. There are two positive constants C and β such that for all γ sufficiently small, $\lambda > 0$ and for all positive locally integrable functions* *f, we have*

$$\omega d\mu_{Q+pt}(\{z\in\mathbb{B}: S_{s+t,s}f(z)>2\lambda, m'_{s+t,s}f(z)\le\gamma\lambda\})\le$$

$$CD_p^{s,t,q,Q}(\omega)\gamma^\beta\omega d\mu_{Q+pt}(\{z\in\mathbb{B}: S_{s+t,s}f(z)>\lambda\}). \quad (7.6.0.17)$$

Proof. Let $\lambda>0$, $0<\gamma<1$ and f a positive locally integrable function. Let $E_\lambda=\{z\in\mathbb{B}: S_{s+t,s}f(z)>\lambda\}$. By the Whitney decomposition Lemma (see [105]), there are a positive integer J, $\delta>1$ and a sequence of pseudo-balls $\{B_j\}_{j=1}^\infty$, with $B_j=B(z_j,r_j)$, such that:

- $E_\lambda=\bigcup_{j=1}^\infty B_j$;
- Every point of E_λ is at most in J balls B_j;
- The balls $B'_j=B(z_j,\delta r_j)$ touch the complement of E_λ in $\mathbb{B}$.

To obtain (7.6.0.17), it is then sufficient to show that

$$\omega\mu_{Q+pt}(\{z\in B: S_{s+t,s}f(z)>2\lambda, m'_{s+t,s}f(z)\le\gamma\lambda\})\le CD_p^{s,t,q,Q}(\omega)\gamma^\beta\omega\mu_{Q+pt}(B) \quad (7.6.0.18)$$

where $B=B(z',r)$ is a ball in the Whitney decomposition of E_λ. From the third property of the Whitney decomposition, there is $z_0\in B'=B(z',\delta r)$ such that $S_{s+t,s}f(z_0)\le\lambda$. Without loss of generallity, assume that there is $\xi_0\in B$ such that $m'_{s+t,s}f(\xi_0)\le\gamma\lambda$. Let $\widetilde{B}=B(z_0,R)$ with $R=\max(1-|z_0|,C_0r)$ where we choose $C_0\ge\max(c_1K(1+\delta),\delta)$ where c_1 is the constant C_1 in Lemma 7.2.15.

We set $f_1=1_{\widetilde{B}}f$ and $f_2=1_{\mathbb{B}\setminus\widetilde{B}}f$, then $f=f_1+f_2$ and by Lemma 7.2.15

and Proposition 7.6.1, we have

$$S_{s+t,s}f_2(z) \leq \int\limits_{\mathbb{B}\setminus\widetilde{B}} \left| \frac{f(\xi)}{|1-\langle z_0,\xi\rangle|^{N+1+s+t}} \right| d\mu_s(\xi) +$$

$$\int\limits_{\mathbb{B}\setminus\widetilde{B}} \left| \frac{1}{|1-\langle z,\xi\rangle|^{N+1+s+t}} - \frac{1}{|1-\langle z_0,\xi\rangle|^{N+1+s+t}} \right| f(\xi) d\mu_s(\xi),$$

so that we finally have

$$S_{s+t,s}f_2(z) \quad \leq \quad S_{s+t,s}f(z_0) + A' m'_{s+t,s}f(\xi_0) \leq \lambda + A'\gamma\lambda.$$

Therefore, to prove (7.6.0.18), it will be enough to show that

$$\omega d\mu_{Q+pt}(\{z \in B : S_{s+t,s}f_1(z) > b\lambda\}) \leq C D_p^{s,t,q,Q}(\omega)\gamma^{\beta}\omega d\mu_{Q+pt}(B). \tag{7.6.0.19}$$

We are going to discuss according to the values of the radius $R = \max(1 - |z_0|, C_0 r)$ of $\widetilde{B} = B(z_0, R)$. Let $E'_\lambda = \{S_{s+t,s}f_1 \geq b\lambda\} \cap B$.

First case: $C_0 r \leq 1 - |z_0|$.

Then, $\widetilde{B} = B(z_0, 1-|z_0|)$. Therefore for all $z \in B$ and $\xi \in \widetilde{B}$, $|1-\langle z,\xi\rangle| \geq 1 - |z| > C'(1-|z_0|)$, so that for all $z \in B$,

$$S_{s+t,s}f_1(z) = \int\limits_{\widetilde{B}} \frac{f(\xi)d\mu_s(\xi)}{|1-\langle z,\xi\rangle|^{N+1+s+t}} \leq \frac{1}{(C'(1-|z_0|))^{N+1+s+t}} \int\limits_{\widetilde{B}} f(\xi)d\mu_s(\xi).$$

Then

$$S_{s+t,s}f_1(z) < C" m'_{s+t,s}f(\xi_0) \leq C"\gamma\lambda.$$

Hence, if we take $0 < \gamma < \gamma_0 = \min(\frac{1}{A'}, \frac{b}{C"})$ then it remains only to prove the following case.

Second case: $1-|z_0|<C_0r$.

Then $\widetilde{B}=B(z_0,C_0r)$ and $E'_\lambda\subseteq L$ for L defined in Proposition 7.6.2. In fact, if $z\in E'_\lambda$, then $z\in B$ and

$$\begin{aligned}
b\lambda\le S_{s+t,s}f_1(z)&=\int_{\widetilde{B}}\frac{f(\xi)d\mu_s(\xi)}{|1-\langle z,\xi\rangle|^{N+1+s+t}}\le\frac{1}{(1-|z|)^{N+1+s+t}}\int_{\widetilde{B}}f(\xi)d\mu_s(\xi)\\
&\le\frac{(C_0r)^{N+1+s+t}}{(1-|z|)^{N+1+s+t}}m'_{s+t,s}f(\xi_0)\\
&\le\frac{(C_0r)^{N+1+s+t}}{(1-|z|)^{N+1+s+t}}\gamma\lambda.
\end{aligned}$$

For $\sigma(z)=R^{s,Q+pt}_{k'}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}}$, with $k'\in(0,\frac{1}{2})$. By Lemma 7.5.24 we have $\sigma\in(A_{p,s+t+\frac{Q-q}{p}})$ so that $\sigma\in(A_{\infty,s+t+\frac{Q-q}{p}})$ because $(A_{p,s+t+\frac{Q-q}{p}})\subseteq(A_{\infty,s+t+\frac{Q-q}{p}})$.

Given the fact that L' is a measurable subset of $\bar{B}=B(z',ar)$, we have by Proposition 7.6.2 and Lemma 7.2.12

$$\begin{aligned}
\omega d\mu_{Q+pt}(L)&\le C\sigma d\mu_{s+t+\frac{Q-q}{p}}(L')\\
&\le C\left(\frac{\mu_{s+t+\frac{Q-q}{p}}(L')}{\mu_{s+t+\frac{Q-q}{p}}(\bar{B})}\right)^{\beta_0}\sigma d\mu_{s+t+\frac{Q-q}{p}}(\bar{B})\\
&\le C\gamma^{\frac{s+t+\frac{Q-q}{p}+1}{N+1+s+t}\beta_0}\sigma d\mu_{s+t+\frac{Q-q}{p}}(\bar{B}).
\end{aligned}$$

As E'_λ is a subset of $L=\{z\in\bar{B}:1-|z|<C'_0\gamma^{\frac{1}{N+1+s+t}}r\}$, it follows that for $\beta=\frac{s+t+\frac{Q-q}{p}+1}{N+1+s+t}\beta_0$ we have:

$$\omega d\mu_{Q+pt}(E'_\lambda)\le C\gamma^{\beta}\sigma d\mu_{s+t+\frac{Q-q}{p}}(\bar{B}).\tag{7.6.0.20}$$

One shows by Fubini's theorem that $\sigma d\mu_{s+t+\frac{Q-q}{p}}(\bar{B})\le C\omega d\mu_{Q+pt}\bar{\bar{B}})$ with $\widetilde{\bar{B}}=B(z',(2k+1)arK)$. And by Corollary 7.5.21 we get $\omega d\mu_{Q+pt}(\widetilde{\bar{B}})\le CD^{s,t,q,Q}_p(\omega)\omega d\mu$

Then

$$\omega d\mu_{Q+pt}(E'_\lambda)=\omega d\mu_{Q+pt}(\{z\in B:S_{s+t,s}f_1(z)>b\lambda\})\le CD^{s,t,q,Q}_p(\omega)\gamma^{\beta}\omega d\mu_{Q+pt}($$

This ends the proof. $\square$

The following results appear as consequence of Theorem 7.6.3 and Lemma 7.2.14.

Theorem 7.6.4. *For $Q \geq q$ and $s+t > -1$, if $\omega \in D_p^{s,t,q,Q}$ there is a constant $C_{s,t,q,Q} > 0$ such that*

$$\int_{\mathbb{B}} (S_{s+t,s} f(z))^p d\mu_{Q+pt}(z) \leq C_{s,t,q,Q} \int_{\mathbb{B}} (m_{s+t,s} f(z))^p d\mu_{Q+pt}(z).$$

Theorem 7.6.5. *For $s+t+\frac{Q-q}{p} > -1$ and $-N-1 < s+t < -1$, if $\omega \in D_p^{s,t,q,Q}$ there is a constant $C_{s,t,q,Q} > 0$ such that*

$$\int_{\mathbb{B}} (S_{s+t,s} f(z))^p d\mu_{Q+pt}(z) \leq C_{s,t,q,Q} \int_{\mathbb{B}} (m'_{s+t,s} f(z))^p d\mu_{Q+pt}(z).$$

7.7 Final remark and open question

This part is simply a direct application of the two preceding sections and Remark 7.1.14. Therefore for $1 < p < +\infty$ we have the two following corollaries.

Corollary 7.7.1. *Let ω be a weight on $\mathbb{B}$. Then for $s+t > -1$ and $q = Q$, the following assertions are equivalent:*

1. *$P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_q)$;*
2. *$T_{s+t,s}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_{q+pt})$;*
3. *$S_{s+t,s}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_{q+pt})$;*
4. *$\omega \in (K_p^{s,t,q,q})$.*

Corollary 7.7.2. *Let ω be a weight on $\mathbb{B}$. In the case both $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$ are hold, and in the case both $s+t > -1$ and $Q \geq q$*

are hold, if $\omega \in D_p^{s,t,q,Q}$ *then* $P_{s,t}$ *is well defined and continuous from* $L^p(\omega d\mu_q)$ *to* $L^p(\omega d\mu_Q)$*, so that* $S_{s+t,s}$ *is well defined and continuous from* $L^p(\omega d\mu_q)$ *to* $L^p(\omega d\mu_{Q+pt})$.

In Theorem 7.5.6 and in Theorem 7.5.7 we show that being in $(K_p^{s,t,q,Q})$ is a necessary condition for the continuity of $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, while in Corollary 7.7.2, we have that being in $D_p^{s,t,q,Q}$ is a sufficient one. When $Q = q$, we find out that $(K_p^{s,t,q,q}) = D_p^{s,t,q,q}$, so that we have a necessary and sufficient condition. But when $Q > q$ we have $D_p^{s,t,q,Q} \subseteq (K_p^{s,t,q,Q})$. It will be interesting in further work to get in that case a necessary and sufficient condition too.

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